On Percolation Connectivity of Large Scale Wireless Networks with Directional Antennas

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***Abstract*—We investigate the *percolation connectivity* of wireless ad hoc networks with directional antennas (called *DIR* networks). One of major concerns is to derive bounds on the number of edge-disjoint directed paths (or *highways*). However, it is non-trivial to obtain bounds on the number of directed highways in *DIR* networks since the conventional *undirected* percolation theory cannot be directly used in *DIR* networks. In this paper, we exploit the *directed* percolation theory to derive bounds on the number of directed highways. In particular, we make new constructions in *bond* directed percolation model. We show that with high probability there are at least $\Omega(\frac{\sqrt{n}}{\log\log\sqrt{n}})$ directed highways in a network with n nodes, which is much tighter than the existing results in *DIR* networks.**

I. Introduction

The connectivity of wireless networks has received extensive research attentions recently [1]–[3] since it is one of the most important measures of the network reliability. Besides, the network connectivity is necessary to ensure the network is fully-connected so that each source node can successfully communicate with its destination node through multi-hop transmissions. In the fully connected wireless networks, it was shown in [4] that each source-destination pair can achieve a throughput of the order $\Theta(\frac{1}{\sqrt{n\log n}})$, where n is the number of nodes in the network. The per-node throughput is significantly decreased as the number of nodes increases. One major reason lies in the *interference* caused by multiple nodes that are simultaneously contending for the same wireless medium. Essentially, the *full-connectivity* of the whole network also pays for the cost of the increased interference as implied in [4], [5]. As suggested in [5], a higher throughput of the order $\Theta(\frac{1}{\sqrt{n}})$ is achievable if the full-connectivity requirement is slightly loosened such that most of nodes in the network are connected by the giant component forming the *backbone highways* (in other words, the network *"percolates"*). Then, the information will be conveyed through the highways in the multi-hop manner. However, most of these studies only consider wireless networks consisting of wireless nodes, each of which is mounted with an *omni-directional* antenna that can cause high interference. We call such wireless networks using omni-directional antennas as *OMN* networks.

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In contrast to an omni-directional antenna, a *directional* antenna can concentrate the radio signal to certain directions so that the interference to other undesired directions can be mitigated. Many recent studies such as [6]–[11] have shown that applying *directional antennas* instead of omni-directional antennas to wireless networks can greatly improve the throughput capacity as well as the network connectivity. We call such wireless networks using directional antennas as *DIR* networks. In particular, a *DIR* network can achieve the per-node throughput of $\Theta(\frac{[G(d)]^2}{\sqrt{n\log n}})$ as shown in [11], where $G(d)$ is the antenna gain factor of a directional antenna. However, most of these studies [6]–[11] only investigate the full-connectivity of the network, which is an *expensive* requirement as shown in previous studies of *OMN* networks [4], [5]. To the best of our knowledge, only one of most recent studies addressed the *percolation connectivity* of *DIR* networks [12]. Specifically, it is shown in [12] that there are $\Omega(\frac{\sqrt{n}}{\log\sqrt{n}})$ highways needed to keep the percolation connectivity, and thus the throughput of a *DIR* network is $\Theta(\frac{[G(d)]^2}{\sqrt{n}\log\sqrt{n}})$. It is obvious that $\Theta(\frac{[G(d)]^2}{\sqrt{n}\log\sqrt{n}}) < \Theta(\frac{[G(d)]^2}{\sqrt{n\log n}})$ since $\log\sqrt{n}$ has the higher order than $\sqrt{\log n}$, implying that the throughput in [12] is even lower than the previous one in [11]. The main reason is that the derived bounds on the number of highways in [12] are not quite tight as suggested from the empirical results in [12]. Therefore, *it is the purpose of this paper to obtain a tighter bound on the number of highways in a DIR network*. However, it is non-trivial to study the percolation connectivity of *DIR* networks since the previous results in *OMN* networks cannot be directly applied to *DIR* networks (e.g., the dual graph [13]).

In this paper, we investigate the *percolate connectivity* of *DIR* networks. We extend the approach in [12] and obtain a tighter bound on the the number of highways in a *DIR* network than that in [12]. In particular, we have the following major findings:

- We find that in a random extended network with n nodes, there are $\Omega(\frac{\sqrt{n}}{\log\log\sqrt{n}})$ highways crossing the network area from left to right. Our derived bounds are tighter than the existing results in [12].

TABLE I
SUMMARY OF OUR WORK

Types of Networks	# of highways in $\mathcal{B}_n$
OMN networks [5]	$\Omega(\sqrt{n})$
DIR networks in [12]	$\Omega(\frac{\sqrt{n}}{\log \sqrt{n}})$
DIR networks in this paper	$\Omega(\frac{\sqrt{n}}{\log\log \sqrt{n}})$

- We also find that those $\Omega(\frac{\sqrt{n}}{\log\log\sqrt{n}})$ highways can be grouped into $O(\frac{\sqrt{n}}{\log\sqrt{n}})$ disjoint sets, each of which has $\Omega(\frac{\log\sqrt{n}}{\log\log\sqrt{n}})$ highways. These highways are almost *straight* and they do not *wiggle* (or swivel) more than $\Omega(\log\sqrt{n})$.

The rest of this paper is organized as follows. We summarize our main findings in Section II. Section III presents the antenna models and the percolation theory. We then derive the main results on the number of highways in Section IV. Finally, Section V concludes this paper.

II. MAIN RESULTS

Before presenting our main results, we first introduce the network model and notations used in this paper. We investigate the connectivity of large scale *DIR* networks in which the number nodes n tends to infinity. There are two types of network settings to achieve this goal: (a) *extended* networks in which the node density is kept constant and the area is increased to infinity; (b) *dense* networks in which the area is kept constant while the node density is increased to infinity. Extended networks and dense networks are widely adopted in the literature [4]–[6], [10]–[12]. Besides, in real-life applications, the results derived under extended networks can be easily applied to dense networks [5] . Thus, we construct a random extended network by placing nodes according to a Poisson point process of unit density (i.e., the density $\lambda = 1$) on the plane. We then consider the network in the square $\mathcal{B}_n = [0, \sqrt{n}] \times [0, \sqrt{n}]$ with area n.

A. Summary of our results

The main goal of this paper is to derive the number of highways in the network area $\mathcal{B}_n$. These highways can form the backbone of the network, which can be used to carry information across the network with a constant rate in the multi-hop manner. In particular, we denote by $N^{\rightarrow}_{x,y(x)}$ the number of edge-disjoint left-to-right directed paths in an area $x \times y(x)$, where $y(x)$ is a function of x. We also denote by $\mathbb{P}\{e\}$ the probability of event e. We then have the following results on $N^{\rightarrow}_{x,y(x)}$.

Theorem 1: For all $\kappa > 0$, there exists $\delta > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}\{N^{\rightarrow}_{\sqrt{n},\kappa\log\sqrt{n}} \leq \frac{\delta\log\sqrt{n}}{\log\log\sqrt{n}}\} = 0.$$

The proof of Theorem 1 will be given in Section IV. Theorem 1 implies that there exist at least $\Omega(\frac{\log\sqrt{n}}{\log\log\sqrt{n}})$ edge-disjoint left-to-right directed paths across an area of $\sqrt{n} \times \kappa\log\sqrt{n}$ and each of these paths will not wiggle more than $\log\sqrt{n}$. Theorem 1 can be easily extended to the whole network area with $\sqrt{n} \times \zeta\sqrt{n}$ ($\zeta > 0$). In particular, we have

Corollary 1: In an extended network with $\sqrt{n} \times \zeta\sqrt{n}$ and $\zeta > 0$, there exists $\xi > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}\{N^{\rightarrow}_{\sqrt{n},\zeta\sqrt{n}} \leq \frac{\xi\sqrt{n}}{\log\log\sqrt{n}}\} = 0.$$

The proof of Corollary 1 will also be given in Section IV. Corollary 1 implies that there exist at least $\Omega(\frac{\sqrt{n}}{\log\log\sqrt{n}})$ edge-disjoint left-to-right directed highways in the whole network. If these highways are used to convey information, then the throughput capacity of a *DIR* network will be $\Omega(\frac{[G(d)]^2}{\sqrt{n}\log\log\sqrt{n}})$, which is even higher than the existing result, i.e $\Theta(\frac{[G(d)]^2}{\sqrt{n}\log n})$ in [11]. Although our results are promising in deriving a higher throughput capacity, it is non-trivial to derive the throughput capacity since it requires many new proving techniques (e.g., *directed* scheduling schemes and *directed* routing schemes), which is left as a future work.

We summarize our results by comparing with the previous results in Table I. As shown in Table I, a *DIR* network with n nodes under our derivation has at least $\Omega(\frac{\sqrt{n}}{\log\log\sqrt{n}})$ highways, which is much higher than that of *DIR* networks derived in [12] and is quite close to that of *OMN*-networks.

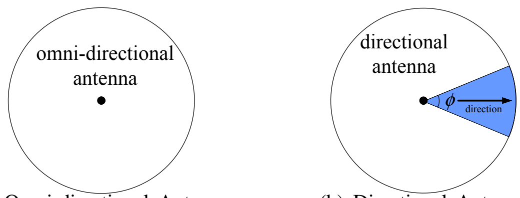
B. Overview of Our Approach

Let us have a glance at our approach used to prove Theorem 1 and Corollary 1. In particular, we construct a mapping from the directional transmissions in *DIR* networks to the *directed bond percolation* model. The connectivity of a *directed lattice* in the directed bond percolation model represents the connectivity of a *DIR* network. Thus, our study is concentrated on the connectivity of directed lattices. Let $m = \sqrt{n}$. To simplify our analysis, we assume that $\log m$, $\log\log m$, $\frac{m}{\log m}$ and $\frac{m}{\log\log m}$ are integers (note that our results can also be easily extended to the non-integer cases like [5]). The *main idea* of our approach is described as the following steps (details will be presented in Section IV):

(1) We construct a directed lattice $L^*_{x,y(x)}$ by using a similar approach to [12], where $*$ means the constructed lattice.
(2) We divide the whole network area into $\frac{m}{\kappa\log m}$ horizontal rectangles, each of which is a directed lattice as constructed in Step (1).
(3) We decompose $L^*_{x,y(x)}$ constructed in Step (1) into a set of directed *sub-lattices*. After the decomposition, we derive the percolation probability of each sub-lattice.
(4) We combine the percolation probability of each sub-lattice together and prove that there exist at least $\frac{\delta\log m}{\log\log m}$ edge-disjoint left-to-right paths in each rectangle constructed in Step (2).

Comparisons between our approach and that in [12]

Our construction in Step (1) is similar to [12]. The key differences between our construction and that of [12] lie in Steps (2) - (4). In particular, we investigate the percolation probability within a rectangle with size $m \times \kappa\log m$ in contrast to that with size $m \times \alpha m$ in [12]. Second, we choose the



(a) Omni-directional Antenna (b) Directional Antenna

Fig. 1. Antennas Models

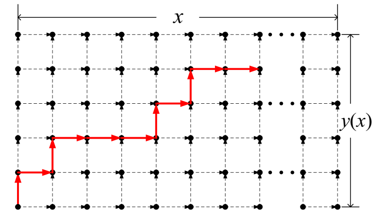


Fig. 2. Directed lattice

different construction of sub-lattices with the smaller portion of *isomeric* bits than that in [12]. As a result, we have the larger number of sub-lattices than that of [12].

III. Models and Preliminaries

In this section, we present the antenna model and some preliminary results in the directed percolation theory.

A. Antenna model

An antenna is a device that is used for radiating/collecting radio signals into/from space. An omni-directional antenna, which can radiate/collect radio signals uniformly in all directions in space, is typically used in conventional wireless ad hoc networks. Different from an omni-directional antenna, a directional antenna can concentrate transmitting or receiving capability to some desired directions so that it has better performance than an omni-directional antenna. To model the transmitting or receiving capability of an antenna, we often use the *antenna gain* to represent the directivity of an antenna. The antenna gain of an omni-directional antenna is assumed to be 1 within 2π since it radiates the radio signal uniformly in all directions, as shown in Fig. 1(a). Similar to the recent studies [11], [12], we assume that the antenna gain of a directional antenna is within a specific angle ϕ (also called the *beamwidth* of the antenna). The antenna gain within ϕ is assumed to $G(d)$ (greater than 1) and the gain outside ϕ is assumed to be 0. At any time, the antenna beam can only be pointed to a certain direction, as shown in Fig. 1(b).

B. Directed Percolation Theory

Generally, we consider a region with area $x \times y(x)$, in which nodes are randomly distributed according to a 2-D Poisson process as assumed in Section II of unit density (density $\lambda = 1$). We then review the directed percolation theory, which will be used to construct the highway systems in *DIR* networks. The undirected percolation theory can be found in [5].

In contrast to *OMN*-networks, there are only directional transmissions in *DIR*-networks in which the undirected percolation theory does not apply. In a *DIR*-network, node X_i can establish a link with node X_j if and only if their antenna-beams cover each other. In this paper, we assume that each node has its neighbor's location information so that it can point its antenna beam to any of its neighbors. Once the antenna beams of the transmitter and the receiver are pointed to each other, they are locked until the transmission is completed [8].

In *DIR*-networks, the directed (oriented) percolation theory has been used to construct highways [12]. We consider a directed lattice $L_{x,y(x)}$ in an area $x \times y(x)$, as shown in Fig. 2. Note that in the directed lattice $L_{x,y(x)}$, the direction of a directed link is either rightwards or upwards. We have the *directed percolation probability* θ_{dir} defined as $\theta_{dir}(p_d) \triangleq \mathbb{P}\{v_0 \to \infty\}$, where $\{v_0 \to \infty\}$ is the event that v_0 is connected to an infinite connected component. Similarly, we denote by p_d^{*bp} the critical threshold of p_d. The formal definition of p_d will be given in Section IV.

As shown in [12], [14], [15], many results of the undirected percolation theory cannot apply for directed lattices. For example, it is shown in [14], [15] the *dual graph* of the undirected percolation theory does not apply for directed lattices. Thus, we have to use some new results from the directed percolation theory. In particular, we show as follows one fundamental result of the directed percolation theory, which will be used to construct the directed highway systems in Section IV.

Lemma 1: [14], [15] The critical threshold in a directed bond model is bounded by $\frac{1}{3} \leq p_d^{*bp} \leq \frac{2}{3}$.

IV. Construction of Highway Systems with Directed Percolation Theory

In this section, we will describe our construction of highway systems in a *DIR* network.

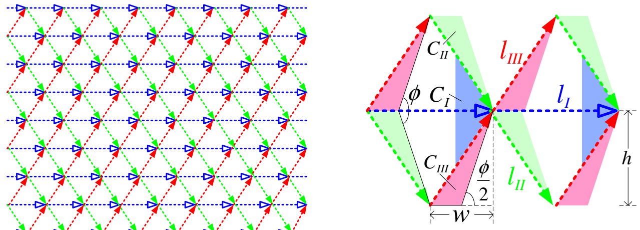
A. Mapping to Bond Directed Percolation Model

In our construction as shown in Fig. 3(a), there are three types of directed links: (1) the *horizontal* link (i.e., the blue links), (2) the *downward slant* link l_{II} (i.e., the green links) and (3) the *upward slant* link (i.e., the red links).

Fig. 3(b) zooms in the construction of the three links. In particular, we consider the antenna beamwidth $\phi \leq \pi$. In our construction (as shown in Fig. 3(b)), there are three types of triangle cells: C_I, C_{II} and C_{III} corresponding to the blue shaded area, the red one and the green one, respectively. Cells C_I, C_{II} and C_{III} contain three types of links - *horizontal* link l_I, *downward slant* link l_{II} and *upward slant* link l_{III}, respectively. A link in a cell is enabled if and only if a node lies in the cell. Similar to [12], we let the area of C_I to be equal to that of C_{II} and that of C_{III}. We denote the area of cells C_I, C_{II} and C_{III} by A_I, A_{II} and A_{III}, respectively. Specifically, we have

$$A = A_I = A_{II} = A_{III} = \frac{h \cdot w}{2} - \frac{1}{2} \cdot h \cdot \frac{h}{\tan \frac{\phi}{2}}$$

where h, w and ϕ are given in Fig. 3(b) and the above equation can be easily obtained with the simple geometry calculation.



(a) Construction of lattice $L^*_{x,y(x)}$ (b) Three types of cells

Fig. 3. Constructions of Directed Lattice

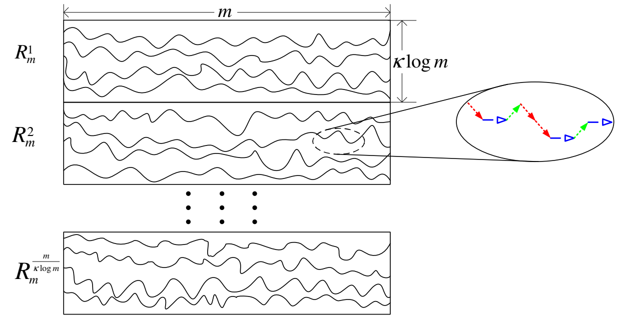


Fig. 4. The network area is divided into $\frac{m}{\kappa \log m}$ horizontal rectangles.

Without loss of generality, we let $\tan\frac{\phi}{2} = \frac{h}{w/2}$. Then, $A = \frac{h\cdot w}{2} - \frac{h\cdot w}{4} = \frac{h\cdot w}{4}$. Thus, we have the probability that a link is enabled as follows: $p_d \triangleq 1 - e^{\frac{1}{4}hw}$.

We appropriately set the *directional transmission range* $r_d \geq \sqrt{4w^2 + h^2}$ such that a node can transmit to any node in its three neighboring cells as shown in Fig. 3(b).

We then divide the whole network area into $\frac{m}{\kappa \log m}$ horizontal rectangles (as shown in Fig. 4), each of which is denoted by R^1_m, R^2_m, $\ldots$ and $R^{\frac{m}{\kappa \log m}}_m$, respectively. Each of the rectangles has a size $m \times \kappa \log m$.

Next, we will prove that we can properly choose p_d and $p_d > p_d^{*bp}$ such that there are $\Omega(\frac{m}{\log\log m})$ paths across the network area from left to right. These paths can be grouped into disjoint sets, each of which has $\Omega(\frac{\log m}{\log\log m})$ paths contained in a rectangle of size $m \times \kappa \log m$.

B. Percolation Probability of Sub-lattices

We first have the two notations for lattice $L^*_{x,y(x)}$ as follows:

- $\{L^{*\rightarrow}_{x,y(x)}\}$: the event that there exists a left-to-right directed path in directed lattice $L^*_{x,y(x)}$.
- $N^{*\rightarrow}_{x,y(x)}$: the number of edge-disjoint left-to-right paths in directed lattice $L^*_{x,y(x)}$.

We then define sub-lattice $L^b_{m,\log(m)}$ of lattice $L^*_{m,\log(m)}$, where we use a bit-string b to specify each of these sub-lattices. In particular, bit-string b represents the orientations of the *slant* links (not *horizontal* links). Note that b has a length of m. Specifically, in sub-lattice $L^b_{m,\log(m)}$, the orientations of slanted links in the j-th rightmost column are specified by the j-th rightmost bit in b. In bit-string b, the '1' in the j-th bit means the *upward slant* links in column j and the '0' in j-th bit means the *downward slant* links in column j. Fig. 5(a) shows an example of sub-lattice $L^b_{m,\log(m)}$ in which there are three ones in bit-string b. Thus, all the slant links in the corresponding columns are upward (i.e., the red links Fig. 5(a)). We then define a sequence of $J = \frac{m}{\log\log m}$ m-bit-strings (notice that J is an integer), denoted by $(b_0, b_1, \cdots, b_J))$, which are shown as follows.

$$\begin{aligned} b_0 &= \quad (\underbrace{000\cdots 00000000}_{m}) \\ b_1 &= \quad (000\cdots 00\,\underbrace{11\cdots 1}_{\log\log m}) \\ &\cdots \\ b_J &= \quad (\underbrace{11\cdots 1}_{\log\log m}\cdots 00000) \end{aligned}$$

where b_i is an all-zero bit-string except for the $(i-1)\log\log m$-th bits to the $i\log\log m$-th bits that are ones and $i = 0, 1, \cdots, J$. We then define a sequence of J directed sub-lattices $(L^{b_0}_{m,\log m}, L^{b_1}_{m,\log m}, \cdots, L^{b_J}_{m,\log m})$, where the i-th sub-lattice is specified by bit-string b_i.

Next, we derive the probability that there exists a left-to-right path in each of the above sub-lattices. Denote by $\{L^{b_i\rightarrow}_{x,y(x)}\}$ the event that there exists a left-to-right directed path in sub-lattice $L^{b_i}_{x,y(x)}$. We then have the following lemma.

Lemma 2: For each bit-string b_i, if $p_d > p_d^{*bp}$, then for any m and for some constant $\epsilon > 0$, we have

$$\mathbb{P}\{L^{b_i\rightarrow}_{m,\log m}\} \geq \epsilon \tag{1}$$

Proof: We present a proof of the lemma in Appendix A. □

We denote by $\mathbb{P}\{L^{*\rightarrow}_{m,\log m}\}$ the probability that there exists a left-to-right directed path in the directed lattice $L^*_{m,\log m}$. We next bound on the probability $\mathbb{P}\{L^{*\rightarrow}_{m,\log m}\}$. We first denote by $\{\overline{L^{b_i\rightarrow}_{m,\log m}}\}$ the complementary event of event $\{L^{b_i\rightarrow}_{m,\log m}\}$. In particular, $\mathbb{P}\{\overline{L^{b_i\rightarrow}_{m,\log m}}\} = 1 - \mathbb{P}\{L^{b_i\rightarrow}_{m,\log m}\}$. We then have the following lemma on the bound of the conditional probability.

Lemma 3: Denote by $\mathbb{P}\{\overline{L^{b_i\rightarrow}_{m,\log m}}|\overline{L^{b_0\rightarrow}_{m,\log m}}, \cdots, \overline{L^{b_{i-1}\rightarrow}_{m,\log m}}\}$ the conditional probability of $\mathbb{P}\{\overline{L^{b_i\rightarrow}_{m,\log m}}\}$ given the events $\{\overline{L^{b_0\rightarrow}_{m,\log m}}, \cdots, \overline{L^{b_{i-1}\rightarrow}_{m,\log m}}\}$. We have

$$\mathbb{P}\{\overline{L^{b_i\rightarrow}_{m,\log m}}|\overline{L^{b_0\rightarrow}_{m,\log m}}, \cdots, \overline{L^{b_{i-1}\rightarrow}_{m,\log m}}\} \leq \mathbb{P}\{\overline{L^{b_i\rightarrow}_{m,\log m}}\} + \frac{\log m \cdot p_d^{\log\log m}}{\mathbb{P}\{\overline{L^{b_{i-1}\rightarrow}_{m,\log m}}\}} \tag{2}$$

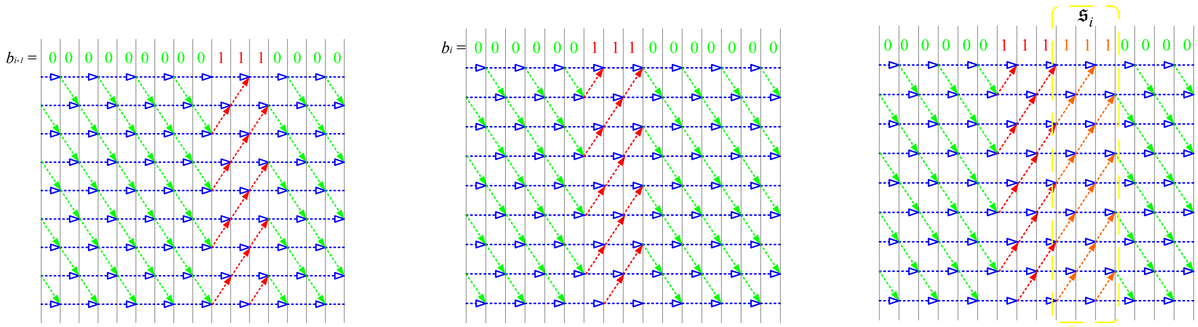
Proof: We present a proof of the lemma in Appendix B. □

Lemma 3 implies that $\mathbb{P}\{\overline{L^{b_i\rightarrow}_{m,\log m}}|\overline{L^{b_0\rightarrow}_{m,\log m}}, \cdots, \overline{L^{b_{i-1}\rightarrow}_{m,\log m}}\} \rightarrow \mathbb{P}\{\overline{L^{b_i\rightarrow}_{m,\log m}}\}$ as $m \rightarrow \infty$. We then apply this result and obtain the following corollary to bound the probability $\mathbb{P}\{L^{*\rightarrow}_{m,\log m}\}$.

Corollary 2: If $p_d > p_{dir}^{*bp}$, $\mathbb{P}\{L^{*\rightarrow}_{m,\log m}\} \rightarrow 1$ as $m \rightarrow \infty$.

Proof: First, we have

$$\begin{aligned} \mathbb{P}\{\overline{L^{*\rightarrow}_{m,\log m}}\} &\leq \mathbb{P}\{\overline{L^{b_0\rightarrow}_{m,\log m}} \cap \cdots \cap \overline{L^{b_J\rightarrow}_{m,\log m}}\} \\ &= \mathbb{P}\{\overline{L^{b_0\rightarrow}_{m,\log m}}\} \cdot \mathbb{P}\{\overline{L^{b_1\rightarrow}_{m,\log m}}|\overline{L^{b_0\rightarrow}_{m,\log m}}\} \cdots \\ &\quad \cdot \mathbb{P}\{\overline{L^{b_J\rightarrow}_{m,\log m}}|\overline{L^{b_0\rightarrow}_{m,\log m}}, \cdots, \overline{L^{b_{J-1}\rightarrow}_{m,\log m}}\} \end{aligned} \tag{3}$$



(a) (b) (c)

Fig. 5. (a) Sub-lattice $L^{b_{i-1}}_{m,\log m}$. (b) Sub-lattice $L^{b_i}_{m,\log m}$. (c) The different part $\mathfrak{s}_i$ between sub-lattices $L^{b_{i-1}}_{m,\log m}$ and $L^{b_i}_{m,\log m}$

Then, by applying the results of Lemma 3 to RHS of Inequality (3), we have

$$\mathbb{P}\{\overline{L^{*\rightarrow}_{m,\log m}}\} \leq \prod_{i=0}^{J}(\mathbb{P}\{\overline{L^{b_i\rightarrow}_{m,\log m}}\} + \frac{\log m \cdot p_d^{\log\log m}}{\mathbb{P}\{L^{b_{i-1}\rightarrow}_{m,\log m}\}}), \tag{4}$$

where $J = \frac{m}{\log\log m}$.

Then, by Lemma 2 and $m \to \infty$, we have

$$\mathbb{P}\{\overline{L^{*\rightarrow}_{m,\log m}}\} \leq (1 - \epsilon + \frac{\log m \cdot p_d^{\log\log m}}{\epsilon})^{\frac{m}{\log\log m}} \to 0 \tag{5}$$

$\square$

C. Proof of Theorem 1 and Corollary 1

We now can prove Theorem 1 by applying Lemmas 2-3 and Corollary 2.

Proof of Theorem 1: As shown in [5], [12], [13], for the number of left-to-right paths in an $x \times y(x)$ lattice, we have that for $p > 0$,

$$\mathbb{P}_p\{N^{\leftrightarrow}_{x,y(x)} \leq \rho\} \leq (\frac{p}{p - p'})^{\rho} \cdot (1 - \mathbb{P}_{p'}\{R^{\leftrightarrow}_{x,y(x)}\}) \tag{6}$$

for any $p > p'$, where $\mathbb{P}_p$ is the probability measure that a link is enabled in the underlying lattice with a probability p. Inequality (6) is based on the property of increasing event $\{R^{\leftrightarrow}_{x,y(x)}\}$ for an undirected lattice $R_{x,y(x)}$ (see [5]). This inequality can be also applied to directed lattice $L^{*}_{x,y(x)}$ since $\{L^{*\rightarrow}_{x,y(x)}\}$ is also an increasing event. In particular, we have that for any $p_d > p'_d$,

$$\mathbb{P}_{p_d}\{N^{\rightarrow}_{x,y(x)} \leq \rho\} \leq (\frac{p_d}{p_d - p'_d})^{\rho} \cdot (1 - \mathbb{P}_{p'_d}\{L^{*\rightarrow}_{x,y(x)}\}) \tag{7}$$

We now consider an $m \times \kappa \log m$ lattice ($\kappa > 1$). Applying to Inequality (7), we then have

$$\mathbb{P}_{p_d}\{N^{\rightarrow}_{m,\kappa\log m} \leq \rho\} \leq (\frac{p_d}{p_d - p'_d})^{\rho} \cdot (1 - \mathbb{P}_{p'_d}\{L^{*\rightarrow}_{m,\kappa\log m}\}) \tag{8}$$

We next generate Lemma 2 as follows.

$$\mathbb{P}_{p_d}\{L^{b_i\rightarrow}_{m,\kappa\log m}\} \geq 1 - (1 - \epsilon)^{\kappa}$$

Similar to the proof of Corollary 2, we have

$$1 - \mathbb{P}_{p'_d}\{L^{*\rightarrow}_{m,\kappa\log m}\} \leq \left((1-\epsilon)^{\kappa} + \frac{\log m \cdot {p'_d}^{\log\log m}}{\epsilon}\right)^{\frac{m}{\log\log m}}$$

Substituting $\rho = \delta\frac{\log m}{\log\log m}$ and $1 - \mathbb{P}_{p'_d}\{L^{*\rightarrow}_{m,\kappa\log m}\}$ into Inequality (8), we have

$$\mathbb{P}_{p_d}\{N^{\rightarrow}_{m,\kappa\log m} \leq \delta\frac{\log m}{\log\log m}\}$$
$$\leq (\frac{p_d}{p_d - p'_d})^{\delta\frac{\log m}{\log\log m}} \cdot \left((1-\epsilon)^{\kappa} + \frac{\log m \cdot {p'_d}^{\log\log m}}{\epsilon}\right)^{\frac{m}{\log\log m}} \tag{9}$$

We can choose p'_d such that $p_d > p'_d > p_d^{*bp}$. Then, when $m \to \infty$, it is always possible to choose a sufficient κ such that the RHS of Inequality (9) approaches 0.

$\square$

We now prove Corollary 1.

Proof of Corollary 1: As shown in Section IV-A, there are $\frac{m}{\kappa\log m}$ horizontal rectangles. Since each rectangle contains at least $\Omega(\frac{\log\sqrt{n}}{\log\log\sqrt{n}})$ edge-disjoint left-to-right highways as proved in Theorem 1), we then combine the bounds on the number of highways in each of the $\frac{\sqrt{n}}{\kappa\log\sqrt{n}}$ rectangles and we then can have the bound on the total number of edge-disjoint left-to-right highways, which is $\Omega(\frac{\sqrt{n}}{\log\log\sqrt{n}})$. $\square$

V. Conclusion

In this paper, we investigate the percolation connectivity of *DIR* networks. In particular, we derive the bounds on the number of highways in *DIR* networks. Specifically, we find that there exist at least $\Omega(\frac{\sqrt{n}}{\log\log\sqrt{n}})$ edge-disjoint paths in a *DIR* network with n nodes and each of these paths will not wiggle too much (more than $\log\sqrt{n}$). Our results can be used to derive the throughput capacity of a *DIR* network, which is left as a future work.

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Appendix A

Proof of Lemma 2: We prove this lemma by using a similar approach to [12]. First, we consider sub-lattice $L^{b_0}_{m,\log m}$. It is obvious that sub-lattice $L^{b_0}_{m,\log m}$ is fully contained in the conventional directed lattice $L_{2m-1,m}$. This is because sub-lattice $L^{b_0}_{m,\log m}$ consists only upward slant links and horizontal links (we can easily obtain this by replacing all the green downward links by the red upward links in Fig. 5(a)). Besides, it is obvious that $\mathbb{P}\{L_{2m-1,m}\} > 0$ implies that $\mathbb{P}\{L^{b_0\to}_{m,\log m}\} > 0$. Then, by Lemma 1, we have $\theta^{bp}_{dir}(p_d) > 0$, i.e., $\mathbb{P}\{L_{2m-1,m}\} > 0$ for any m when $p_d > p^{*bp}_{dir}$.

We then consider sub-lattice $L^{b_i}_{m,\log m}$ where $i > 0$ and $i \le J$. There are $\log\log m$ bits different from b_0, which only counts for a very small portion among m bits. When $m \to \infty$, the limitation of the fraction $\lim_{m\to\infty} \frac{\log\log m}{m} = 0$, implying that the small portion of different bits has no impacts on the percolation. Therefore, we have $\mathbb{P}\{L^{b_i\to}_{m,\log m}\} > 0$. □

Appendix B

Proof of Lemma 3: We also use a similar approach to [12] to prove the result though we have a more rigid proof in this paper compared with [12]. We denote by sub-lattice $L^{\mathfrak{s}_i}_{\log\log m,\log m}$ in which the first $\log\log m$ columns that sub-lattices $L^{b_i}_{m,\log m}$ and $L^{b_{i-1}}_{m,\log m}$ differ, where $\mathfrak{s}_i$ is the first $\log\log m$ bits that bits b_i and b_{i-1} differ (we count bits from the rightmost). Figs. 5 (a), (b) and (c) show sub-lattices $L^{b_i}_{m,\log m}$, $L^{b_{i-1}}_{m,\log m}$ and $\mathfrak{s}_i$, respectively. We then let Y the event that there exists a row of $\log\log m$ horizontal enabled links in sub-lattice $L^{\mathfrak{s}_i}_{\log\log m,\log m}$. We also denote by $\overline{Y}$ the complimentary event of event Y. We then have

$$\begin{aligned}
&\mathbb{P}\{L^{b_i\to}_{m,\log m}|L^{b_{i-1}\to}_{m,\log m}\} \\
=\;& \mathbb{P}\{L^{b_i\to}_{m,\log m}\cap\overline{Y}|L^{b_{i-1}\to}_{m,\log m} + \mathbb{P}\{L^{b_i\to}_{m,\log m}\cap Y|L^{b_{i-1}\to}_{m,\log m}\} \\
\le\;& \mathbb{P}\{L^{b_i\to}_{m,\log m}\cap\overline{Y}|L^{b_{i-1}\to}_{m,\log m}\cap\overline{Y} + \mathbb{P}\{L^{b_i\to}_{m,\log m}\cap Y|L^{b_{i-1}\to}_{m,\log m}\}
\end{aligned} \tag{10}$$

In the last inequality of Eq. (10), we let $\mathbb{C}_1 = \mathbb{P}\{L^{b_i\to}_{m,\log m}\cap\overline{Y}|L^{b_{i-1}\to}_{m,\log m}\cap\overline{Y}\}$ and $\mathbb{C}_2 = \mathbb{P}\{L^{b_i\to}_{m,\log m}\cap Y|L^{b_{i-1}\to}_{m,\log m}\}$. We next analyze component $\mathbb{C}_1$ and component $\mathbb{C}_2$ as follows.

Component $\mathbb{C}_1$:

Suppose that there exists no such a row of horizontal enabled links in sub-lattice $L^{\mathfrak{s}_i}_{\log\log m,\log m}$, then any path in sub-lattice $L^{b_{i-1}}_{m,\log m}$ must traverse at least one upward slant links in sub-lattice $L^{\mathfrak{s}_i}_{\log\log m,\log m}$ and any path in sub-lattice $L^{b_i}_{m,\log m}$ must traverse at least one downward slant links in sub-lattice $L^{\mathfrak{s}_i}_{\log\log m,\log m}$. Therefore, events $\{L^{b_{i-1}\to}_{m,\log m}\}$ and $\{L^{b_i\to}_{m,\log m}\}$ are independent under event $\overline{Y}$ since the paths in the two sub-lattices must traverse at least one different links. Thus, we have

$$\mathbb{P}\{L^{b_i\to}_{m,\log m}\cap\overline{Y}|L^{b_{i-1}\to}_{m,\log m}\cap\overline{Y}\} = \mathbb{P}\{L^{b_i\to}_{m,\log m}\cap\overline{Y}\} \le \mathbb{P}\{L^{b_i\to}_{m,\log m}\}$$

Component $\mathbb{C}_2$:

Suppose that there exists a row of horizontal enabled links in sub-lattice $L^{\mathfrak{s}_i}_{\log\log m,\log m}$. Notice that sub-lattices $L^{b_{i-1}}_{m,\log m}$ and $L^{b_i}_{m,\log m}$ are identical with the exclusion of sub-lattice $L^{\mathfrak{s}_i}_{\log\log m,\log m}$. Hence, events $\{L^{b_{i-1}\to}_{m,\log m}\}$ and $\{L^{b_i\to}_{m,\log m}\}$ are positively correlated. Then, we have

$$\mathbb{P}\{L^{b_i\to}_{m,\log m}\cap Y|L^{b_{i-1}\to}_{m,\log m}\} \le \mathbb{P}\{Y|L^{b_{i-1}\to}_{m,\log m}\} \le \mathbb{P}\{Y\}/\mathbb{P}\{L^{b_{i-1}\to}_{m,\log m}\}$$

Notice that $\mathbb{P}\{Y\} \le \log m \cdot p_d^{\log\log m}$, which is obtained by the union bound of $\log m$ rows and the probability that all $\log\log m$ horizontal links in a row are enabled is $p_d^{\log\log m}$.

After integration of the analysis of component $\mathbb{C}_1$ and component $\mathbb{C}_2$, we have

$$\mathbb{P}\{L^{b_i\to}_{m,\log m}|L^{b_{i-1}\to}_{m,\log m}\} \le \mathbb{P}\{L^{b_i\to}_{m,\log m}\} + \frac{\log m\cdot p_d^{\log\log m}}{\mathbb{P}\{L^{b_{i-1}\to}_{m,\log m}\}} \tag{11}$$

To simplify the following analysis, we define $\frac{\log m\cdot p_d^{\log\log m}}{\mathbb{P}\{L^{b_{i-1}\to}_{m,\log m}\}}$ as $f(m)$. Then, Inequality (11) can be rewritten as

$$\mathbb{P}\{L^{b_i\to}_{m,\log m}|L^{b_{i-1}\to}_{m,\log m}\} \le \mathbb{P}\{L^{b_i\to}_{m,\log m}\} + f(m) \tag{12}$$

We then times $\mathbb{P}\{L^{b_{i-1}\to}_{m,\log m}\}$ to LHS and RHS of Inequality (12), respectively. Then we have

$$\mathbb{P}\{L^{b_i\to}_{m,\log m}\cap L^{b_{i-1}\to}_{m,\log m}\} \le \mathbb{P}\{L^{b_{i-1}\to}_{m,\log m}\}\cdot(\mathbb{P}\{L^{b_i\to}_{m,\log m}\} + f(m)) \tag{13}$$

Since $\mathbb{P}\{\overline{L^{b_{i-1}\to}_{m,\log m}}\} = 1 - \mathbb{P}\{L^{b_{i-1}\to}_{m,\log m}\}$, then Inequality (13) can be rewritten as

$$\mathbb{P}\{L^{b_i\to}_{m,\log m}\cap L^{b_{i-1}\to}_{m,\log m}\} \le (1-\mathbb{P}\{\overline{L^{b_{i-1}\to}_{m,\log m}}\})\cdot(\mathbb{P}\{L^{b_i\to}_{m,\log m}\}+f(m)) \tag{14}$$

Thus, we have

$$\begin{aligned}
&\mathbb{P}\{L^{b_i\to}_{m,\log m}\cap L^{b_{i-1}\to}_{m,\log m}\} + \mathbb{P}\{\overline{L^{b_{i-1}\to}_{m,\log m}}\} + \mathbb{P}\{\overline{L^{b_i\to}_{m,\log m}}\} - 1 \\
\le\;& \mathbb{P}\{\overline{L^{b_{i-1}\to}_{m,\log m}}\}\cdot\mathbb{P}\{\overline{L^{b_i\to}_{m,\log m}}\} + f(m)\cdot\mathbb{P}\{L^{b_{i-1}\to}_{m,\log m}\}
\end{aligned}$$

That is

$$\mathbb{P}\{\overline{L^{b_i\to}_{m,\log m}}\cap\overline{L^{b_{i-1}\to}_{m,\log m}}\} \le \mathbb{P}\{\overline{L^{b_{i-1}\to}_{m,\log m}}\}\cdot\mathbb{P}\{\overline{L^{b_i\to}_{m,\log m}}\}+f(m)\cdot\mathbb{P}\{L^{b_{i-1}\to}_{m,\log m}\} \tag{15}$$

Without loss of generality, we assume that $\mathbb{P}\{L^{b_{i-1}\to}_{m,\log m}\} \le \mathbb{P}\{\overline{L^{b_{i-1}\to}_{m,\log m}}\}$. Dividing LHS and RHS of Inequality (15) by $\mathbb{P}\{\overline{L^{b_{i-1}\to}_{m,\log m}}\}$, respectively, we then have

$$\mathbb{P}\{\overline{L^{b_i\to}_{m,\log m}}|\overline{L^{b_{i-1}\to}_{m,\log m}}\} \le \mathbb{P}\{\overline{L^{b_i\to}_{m,\log m}}\} + \frac{\log m\cdot p_d^{\log\log m}}{\mathbb{P}\{L^{b_{i-1}\to}_{m,\log m}\}} \tag{16}$$

We then apply the similar argument with consideration of the first $\log\log m$ bits that b_i and b_j differ ($j < i$). After that, we have the above results. □