IS-LABEL: an Independent-Set based Labeling Scheme for Point-to-Point Distance Querying

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ABSTRACT

We study the problem of computing shortest path or distance between two query vertices in a graph, which has numerous important applications. Quite a number of indexes have been proposed to answer such distance queries. However, all of these indexes can only process graphs of size barely up to 1 million vertices, which is rather small in view of many of the fast-growing real-world graphs today such as social networks and Web graphs. We propose an efficient index, which is a novel labeling scheme based on the independent set of a graph. We show that our method can handle graphs of size orders of magnitude larger than existing indexes.

1. INTRODUCTION

Computing the shortest path or distance between two vertices is a basic operation in processing graph data. The importance of the operation is not only because of its role as a key building block in many algorithms but also of its numerous applications itself. In addition to applications in transportation, VLSI design, urban planning, operations research, robotics, etc., the proliferation of network data in recent years has introduced a broad range of new applications. For example, social network analysis, page similarity measurement in Web graphs, entity relationship ranking in semantic Web ontology, routing in telecommunication networks, context-aware search in social networking sites, to name but a few.

In many of these new applications, however, the size of the underlying graph is often in the scale of millions to billions of vertices and edges. Such large graphs are becoming more and more common, some of the well-known ones include Web graphs, various social networks (e.g., Twitter, Facebook, LinkedIn), RDF graphs, mobile phone networks, SMS networks, etc. Computing shortest path or distance in these large graphs with conventional algorithms such as Dijkstra's algorithm or simple BFS may result in a long running time that is not acceptable.

For computing shortest path or distance between two points in a road network, many efficient indexes have been proposed [1, 2, 4, 9, 16, 17, 30, 31, 32]. However, these works apply unique properties of road networks and hence are not applicable for other graphs/networks that are not similar to road networks. In recent years, a number of indexes have been proposed to process distance queries in general sparse graphs [12, 15, 16, 22, 34, 38, 40]. However, as we will discuss in details in Section 7, these indexes can only handle relatively small graphs due to high index construction cost and large index storage space. As a reference, the largest real graphs tested in these works have only 581K vertices with average degree 2.45 [12], and 694K vertices with average degree 0.45 [22], while most of the other real graphs tested are significantly smaller.

We propose a new index for computing shortest path or distance between two query vertices and our method can handle graphs with hundreds of millions of vertices and edges. Our index, named as **IS-Label**, is designed based on a novel application of the *independent set* of a graph, which allows us to organize the graph into layers that form a hierarchical structure. The hierarchy can be used to guide the shortest path computation and hence leads to the design of effective vertex labels (i.e., the index) for distance computation.

We highlight the main contributions of our paper as follows.

- We propose an efficient index for answering shortest path or distance queries, which can handle graphs up to orders of magnitude larger than those tested in the existing works [12, 15, 16, 22, 34, 38, 40]. None of these existing works can handle even the medium-sized graphs that we tested.
- We design an effective labeling scheme such that the label size remains small even if no optimization (mostly NP-hard) is applied as in the existing labeling schemes.
- Our index naturally lends itself to the design of simple and efficient algorithms for both index construction and query processing.
- We develop I/O-efficient algorithms to construct vertex labels in large graphs that cannot fit in main memory.
- We verify both the efficiency and scalability of our method for processing distance queries in large real-world graphs.

Organization. Section 2 defines the problem and basic notations. Sections 3 and 4 present the details of index design, and Section 5 describes the algorithms. Section 6 reports the experimental results. Section 7 discusses the limitations of existing works. Section 8 discusses handling path queries and Section 9 concludes the paper.

2. NOTATIONS

We focus our discussion on weighted, undirected simple graphs. Let $G = (V_G, E_G, \omega_G)$ be such a graph, where V_G is the set of vertices, E_G is the set of edges, and $\omega_G : E_G \rightarrow \mathbb{N}^+$ is a function that assigns to each edge a positive integer as its weight. We denote

the weight of an edge (u, v) by $\omega(u, v)$. The size of G is defined as $|G| = (|V_G| + |E_G|)$.

We define the set of *adjacent* vertices (or *neighbors*) of a vertex v in G as $adj_G(v) = \{u : (u, v) \in E_G\}$, and the *degree* of v in G as $deg_G(v) = |adj_G(v)|$.

We assume that a graph is stored in its adjacency list representation, where each vertex is assigned a unique vertex ID and vertices are ordered in ascending order of their vertex IDs.

Given a path p in G, the *length* of p is defined as $len(p) = \sum_{e \in p} \omega_G(e)$, i.e., the sum of the weights of the edges on p. Given two vertices $u, v \in V_G$, the *shortest path* from u to v, denoted by $SP_G(u, v)$, is a path in G that has the minimum length among all paths from u to v in G. We define the *distance* from u to v in G as $dist_G(u, v) = len(SP_G(u, v))$. We define $dist_G(v, v) = 0$ for any $v \in V_G$.

Problem definition: we study the following problem: given a *static graph* $G = (V_G, E_G, \omega_G)$, construct a *disk-based index* for processing *point-to-point* (**P2P**) shortest path distance queries, i.e., given any pair of vertices $(s, t) \in (V_G \times V_G)$, find $dist_G(s, t)$.

We focus on *sparse graphs*, which are prevalent in real world. We focus our discussion on undirected graphs, and will also show that our index can be extended to handle directed graphs in Section 4.3. We will also discuss computing the actual path in Section 8, which is a fairly simple extension with some extra bookkeeping.

3. QUERYING DISTANCE BY VERTEX HIERARCHY

In this section, we present our main indexing scheme, which consists of the following components:

- A layered structure of vertex hierarchy constructed from the input graph.
- A vertex labeling scheme developed from the vertex hierarchy.
- Query processing using the set of vertex labels.

We discuss each of these three components in Sections 3.1 to 3.3.

3.1 Construction of Vertex Hierarchy

The main idea of our index is to assign hierarchy to vertices in an input graph G so that we can use the vertex hierarchy to compute the vertex labels, which are then used for querying distance.

To create hierarchies for vertices in G, we construct a layered hierarchical structure from G. To formally define the hierarchical structure, we first need to define the following two important properties that are crucial in the design of our index:

- **Vertex independence**: given a graph $H = (V_H, E_H, \omega_H)$, and a set of vertices I, we say that I maintains the vertex independence property with respect to H if $I \subseteq V_H$ and $\forall u, v \in I$, $(u, v) \notin E_H$, i.e., I is an *independent set* of H.
- **Distance preservation**: given two graphs $H_1 = (V_{H_1}, E_{H_1}, \omega_{H_1})$ and $H_2 = (V_{H_2}, E_{H_2}, \omega_{H_2})$, we say that H_2 maintains the distance preservation property with respect to H_1 if $\forall u, v \in V_{H_2}$, $dist_{H_2}(u, v) = dist_{H_1}(u, v)$.

While distance preservation is essential for processing distance queries, vertex independence is critical for efficient index construction as we will see later when we introduce the index.

We now formally define the layered hierarchical structure, followed by an illustrating example.

DEFINITION 1 (VERTEX HIERARCHY). *Given a graph $G = (V_G, E_G, \omega_G)$, a **vertex hierarchy** structure of G is defined by a pair $(\mathbb{L}, \mathbb{G})$, where $\mathbb{L} = \{L_1, \ldots, L_h\}$ is a set of vertex sets and $\mathbb{G} = \{G_1, \ldots, G_h\}$ is a set of graphs such that:*

- *$V_G = L_1 \cup \ldots \cup L_h$, and $L_i \cap L_j = \emptyset$ for $1 \leq i < j \leq h$;*
- *For $1 \leq i \leq h$, each L_i maintains the vertex independence property with respect to G_i, i.e., L_i is an independent set of G_i;*
- *$G_1 = G$, and for $2 \leq i \leq h$, let $G_i = (V_{G_i}, E_{G_i}, \omega_{G_i})$, then $V_{G_i} = (V_G - L_1 - ... - L_{i-1})$, whereas E_{G_i} and ω_{G_i} satisfy the condition that G_i maintains the distance preservation property with respect to G_{i-1}.*

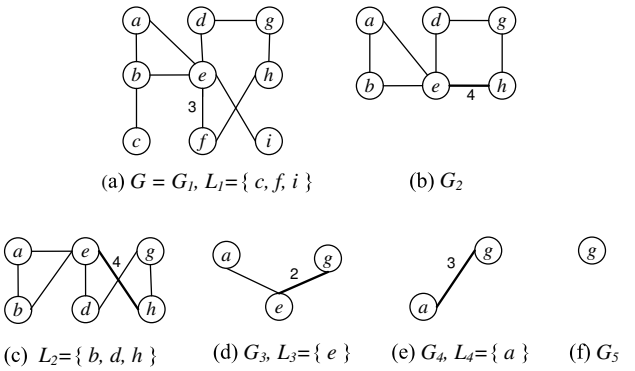
Intuitively, $\mathbb{L}$ is a *partition* of the vertex set V_G and represents a vertex hierarchy, where L_i is at a lower hierarchical level than L_j for $i < j$. Meanwhile, each $G_i \in \mathbb{G}$ preserves the distance information in the original graph G, as shown by the following lemma.

LEMMA 1. *For all $u, v \in V_{G_i}$, where $1 \leq i \leq h$, $dist_{G_i}(u, v) = dist_G(u, v)$.*

PROOF. Since for any $u, v \in V_{G_i}$, $u, v \in V_{G_j}$ for $1 \leq j \leq i$. Thus, we have $dist_{G_i}(u, v) = dist_{G_{i-1}}(u, v) = \ldots = dist_{G_1}(u, v) = dist_G(u, v)$ since each G_i maintains the distance preservation property with respect to G_{i-1} for $2 \leq i \leq h$. □

We use the following example to illustrate the concept of vertex hierarchy.

EXAMPLE 1. *Figure 1 shows a given graph G and the vertex hierarchy of G. We assume that each edge in G has unit weight except for (e, f), which has a weight of 3. It is obvious that the set $\{c, f, i\}$ forms an independent set in G, similarly $\{b, d, h\}$ in G_2 and $\{e\}$ in G_3. It is easy to see that G_2 preserves all distances in G, we shall explain the addition of edge (e, h) later. In order to preserve the distance in G_2, an edge (e, g) of weight 2 is added to G_3. G_4 consists of a single edge (a, g) of weight 3. $L_4 = \{a\}$, G_5 consists of a single vertex g, $L_5 = \{g\}$.*



(a) $G = G_1$, $L_1 = \{ c, f, i \}$ (b) G_2

(c) $L_2 = \{ b, d, h \}$ (d) G_3, $L_3 = \{ e \}$ (e) G_4, $L_4 = \{ a \}$ (f) G_5

Figure 1: A vertex hierarchy

The distance preservation property can be maintained in G_i with respect to G_{i-1} as follows. First, we require the subgraph of G_{i-1} induced by the vertex set V_{G_i} to be in G_i (i.e. $(u, v) \in E_{G_i}$ iff $(u, v) \in E_{G_{i-1}}$ for $u, v \in V_{G_i}$). Then, we create a set of additional edges, called **augmenting edges**, to be included into E_{G_i} as

follows. For any vertex $v \in L_{i-1}$ (thus $v \notin V_{G_i}$ according to Definition 1), if $u, w \in V_{G_i}$, $(u, v) \in E_{G_{i-1}}$ and $(v, w) \in E_{G_{i-1}}$, then an augmenting edge (u, w) is created in G_i with $\omega_{G_i}(u, w) = \omega_{G_{i-1}}(u, v) + \omega_{G_{i-1}}(v, w)$. If (u, w) already exists in G_i, then $\omega_{G_i}(u, w) = \min(\omega_{G_{i-1}}(u, w), \omega_{G_{i-1}}(u, v) + \omega_{G_{i-1}}(v, w))$. An edge in G_i with updated weight is also called an augmenting edge. For example, in Figure 1, in G_3, $dist(e, g)$ can be preserved by creating an augmenting edge (e, g) with $\omega(e, g) = 2$. Edge (e, h) is also added according to our process above. Note that $dist_{G_1}(e, h) = 3$, which can be preserved in G_2 without adding (e, h), but we leave (e, h) there to avoid costly distance querying needed to exclude (e, h).

The following lemma shows the correctness of constructing G_i from G_{i-1} as discussed above.

LEMMA 2. *Constructing G_i from G_{i-1}, where $2 \leq i \leq h$, by adding augmenting edges to the induced subgraph of G_{i-1} by V_{G_i}, maintains the distance preservation property with respect to G_{i-1}.*

PROOF. According to Definition 1, L_{i-1} is the only set of vertices that are in G_{i-1} but missing in G_i. For any two vertices s and t in G_i, suppose that the shortest path (in G_{i-1}) from s to t, $SP_{G_{i-1}}(s, t)$ does not pass through any vertex in L_{i-1}, then the distance between s and t in G_{i-1} is trivially preserved in G_i. Next suppose $SP_{G_{i-1}}(s, t)$ passes through some vertex $v \in L_{i-1}$. Let $SP_{G_{i-1}}(s, t) = \langle s, \ldots, u, v, w, \ldots, t \rangle$. Then, we must have the augmenting edge (u, w) created in G_i with $\omega_{G_i}(u, w) = \omega_{G_{i-1}}(u, v) + \omega_{G_{i-1}}(v, w)$, or $\omega_{G_i}(u, w) = \min(\omega_{G_{i-1}}(u, w), \omega_{G_{i-1}}(u, v) + \omega_{G_{i-1}}(v, w))$ if (u, w) already exists in G_i. Therefore, the distance (in G_{i-1}) between any two vertices is preserved in G_i. □

In addition to the distance preservation property that is required for answering distance queries, the proof also gives a hint on why we require each L_i to be an independent set of G_i. Since there is no edge in G_{i-1} between any two vertices in L_{i-1}, to create an augmenting edge (u, w) in G_i we only need to do a self-join on the neighbors of the vertex $v \in L_{i-1}$. Thus, the search space is limited to 2 hops from each vertex. On the contrary, if an edge can exist between two vertices in L_{i-1}, then to preserve the distance the search space is at least 3 hops from each vertex, which is significantly larger than the 2-hop search space in practice. This is crucial for processing a large graph that cannot fit in main memory as we may need to scan the graph many times to perform the join, as we will see in Section 5.

3.2 Vertex Labeling

With the vertex hierarchy $(\mathbb{L}, \mathbb{G})$, we now describe a labeling scheme that can facilitate fast computation of P2P distance. We first define the following concepts necessary for the labeling.

- **Level number**: each vertex $v \in V_G$ is assigned a level number, denoted by $\ell(v)$, which is defined as $\ell(v) = i$ iff $v \in L_i$.
- **Ancestor**: a vertex $u \in V_G$ is an ancestor of a vertex v if there exists a sequence $S = \langle v = w_1, w_2, ..., w_p = u \rangle$, such that $\ell(w_1) < \ell(w_2) < ... < \ell(w_p)$, and for $1 \leq i < p$, the edge $(w_i, w_{i+1}) \in E_{G_j}$ where $j = \ell(w_i)$. Note that v is an ancestor of itself. If u is an ancestor of v, then v is a **descendant** of u.

EXAMPLE 2. *In our example in Figure 1, the level numbers of c, f, i are 1, that of b, d, h are 2, that of e is 3. The ancestors of f will be e, h, a, g, since (f, e) and (f, h) are in G_1, (h, g) is in G_2, and (e, a), (e, g) are in G_3. Note that d is not an ancestor of f since in the path $\langle f, e, d \rangle$, $\ell(e) = 3$ while $\ell(d) = 2$. The ancestor-descendant relationships are shown in Figure 2(a).*

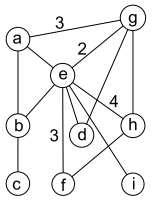
We now define vertex label as follows.

DEFINITION 2 (VERTEX LABEL). *The **label** of a vertex $v \in V_G$, denoted by $LABEL(v)$, is defined as $LABEL(v) = \{(u, dist_G(v, u)) : u \in V_G$ is an ancestor of $v\}$.*

To compute $LABEL(v)$ for all $v \in V_G$, we need to compute the distance from v to each of v's ancestors. This is an expensive process which cannot be scaled to process large graphs. To address this problem, we define a relaxed vertex label that requires only an upper-bound, $d(v, u)$, of $dist_G(v, u)$ and show that $d(v, u)$ suffices for answering distance queries.

DEFINITION 3 (RELAXED VERTEX LABEL). *The **relaxed label** of a vertex $v \in V_G$, denoted by $label(v)$, is a set of "$(u, d(v, u))$" pairs computed by the following procedure: For each $v \in V_G$, we first include $(v, 0)$ in $label(v)$ and* mark *v. Then, we add more entries to $label(v)$ recursively as follows. Take a* marked *vertex u that has the smallest level number $\ell(u)$, and* unmark *u. Let $\ell(u) = j$. For each $w \in adj_{G_j}(u)$, where $\ell(w) > j$ and $(w, d(v, w)) \notin label(v)$, add the entry $(w, (d(v, u) + \omega_{G_j}(u, w)))$ to $label(v)$, and* mark *w. If the entry $(w, d(v, w))$ is already in $label(v)$, update $d(v, w) = \min(d(v, w), (d(v, u) + \omega_{G_j}(u, w)))$. Repeat the above recursive process until no more vertex is marked.*

As for $LABEL(v)$, $label(v)$ contains entries for all ancestors of v. In Section 5, we will show that the new definition facilitates the design of an I/O-efficient algorithm for handling large graphs. Here, we further illustrate the concept using an example, and then prove that $label(v)$ can indeed be used instead of $LABEL(v)$ to correctly answer P2P distance queries in the following subsection.



(a)

$label(c)$	$\{(a, 2), (b, 1), (c, 0), (e, 2), (g, 4)\}$
$label(f)$	$\{(a, 4), (e, 3), (f, 0), (g, 5), (h, 1)\}$
$label(i)$	$\{(a, 2), (e, 1), (g, 3), (i, 0)\}$
$label(b)$	$\{(a, 1), (b, 0), (e, 1), (g, 3)\}$
$label(d)$	$\{(a, 2), (d, 0), (e, 1), (g, 1)\}$
$label(h)$	$\{(a, 5), (e, 4), (g, 1), (h, 0)\}$
$label(e)$	$\{(a, 1), (e, 0), (g, 2)\}$
$label(a)$	$\{(a, 0), (g, 3)\}$
$label(g)$	$\{(g, 0)\}$

(b)

Figure 2: Labeling for the example in Figure 1

EXAMPLE 3. *For our example in Figure 1, the ancestor relationships are shown in Figure 2(a), where all edges have unit weights unless indicated otherwise. The labeling starts with L_1, for vertices c, f, i, next L_2 vertices b, d, h are labeled, followed by $L_3 = \{e\}$, $L_4 = \{a\}$, and $L_5 = \{g\}$. Consider the labeling for vertex c, first, $(c, 0)$ is included, since $adj_G(c) = \{b\}$, $(b, 1)$ is added to $label(c)$ and b is marked. b is unmarked by checking its neighbors a and e in G_2, and we include both $(a, 2), (e, 2)$ into $label(c)$, a and e are marked. e is at level 3 and is unmarked next. $adj_{G_3}(e) = \{a, g\}$, we add $(g, 4)$ to $label(c)$. Then a is unmarked, its only neighbor g in G_4 is already in $label(c)$, $d(c, g)$ is not updated. g is marked. Finally g is unmarked, since g has no neighbor in G_5, no further processing is required. The labels for all vertices are shown in Figure 2(b). Note that $d(h, e) = 4$ in $label(h)$, while $dist_G(h, e) = 3$, hence $d(h, e) > dist_G(h, e)$. In general the distance value in a label entry can be greater than the true distance.*

3.3 P2P Distance Querying

We now discuss how we use the vertex labels to answer P2P distance queries. We first define the following label operations used in query processing.

- **Vertex extraction**: $\mathcal{V}[label(v)] = \{u : (u, d(v,u)) \in label(v)\}$.
- **Label intersection**: $label(u) \cap label(v) = \mathcal{V}[label(u)] \cap \mathcal{V}[label(v)]$.

The above two operations apply in the same way to $LABEL(.)$.

Given a P2P distance query with two input vertices, s and t, let $\mathbb{X} = label(s) \cap label(t)$, the query answer is given as follows.

$$dist_G(s,t) = \begin{cases} \min_{w \in \mathbb{X}}\{d(s,w) + d(w,t)\} & \text{if } \mathbb{X} \neq \emptyset \\ \infty & \text{if } \mathbb{X} = \emptyset \end{cases} \quad (1)$$

In Equation 1, we retrieve $d(s,w)$ and $d(t,w)$ for each $w \in \mathbb{X}$ from $label(s)$ and $label(t)$, respectively. We give an example of answering P2P distance queries using the vertices labels as follows.

EXAMPLE 4. *Consider the example in Figure 1, the labeling is shown in Figure 2. Suppose we are interested in $dist_G(h,e)$. We look up $label(h)$ and $label(e)$. $label(h) \cap label(e) = \{e, a, g\}$. Among these vertices, g has the smallest sum of $d(h,g)+d(g,e) = 1+2 = 3$. Hence we return 3 as $dist_G(h,e)$. Note that although the distance $d(h,e)$ recorded in $label(h)$ is 4, which is greater than $dist_G(h,e)$, the correct distance is returned. If we want to find $dist_G(a,g)$, $label(a) \cap label(g) = \{g\}$. Hence $dist_G(a,g)$ is given by $d(a,g) + d(g,g) = 3 + 0 = 3$.*

Query processing using the vertex labels is simple; however, it is not straightforward to see how the answer obtained is correct for every query. In the remainder of this section, we prove the correctness of the query answer obtained using the vertex labels.

We first define the concept of **max-level vertex**, denoted by v_{max}, of a shortest path, which is useful in our proofs. Given a shortest path from s to t in G, $SP_G(s,t) = \langle s = v_1, v_2, \ldots, v_p = t\rangle$, v_{max} is the max-level vertex of $SP_G(s,t)$ if v_{max} is a vertex on $SP_G(s,t)$ and $\ell(v_{max}) \geq \ell(v_i)$ for $1 \leq i \leq p$. The following lemma shows that v_{max} is unique in any shortest path.

LEMMA 3. *Given two vertices s and t, if s and t are connected in G, then for any shortest path $SP_G(s,t)$ between s and t, there exists a unique max-level vertex, v_{max}, of $SP_G(s,t)$.*

PROOF. If s and t are connected in G, then at least one shortest path exists between s and t. Consider any such shortest path $SP_G(s,t)$, v_{max} must exist on $SP_G(s,t)$. Now suppose to the contrary that v_{max} is not unique, i.e., there exists at least one other vertex v on $SP_G(s,t)$ such that $\ell(v_{max}) = \ell(v) = j$, which also means that both v_{max} and v are in L_j and G_j. Since L_j is an independent set of G_j, there is no edge between v_{max} and v in G_j. Since v_{max} and v are on the same path $SP_G(s,t)$, they must be connected in G_j and the path connecting them must pass through some neighbor u of v_{max} or v in G_j, where u is also on $SP_G(s,t)$. Thus, u cannot be in L_j (otherwise the vertex independence property is violated) and hence $\ell(u) > \ell(v_{max})$, which contradicts that v_{max} is the max-level vertex of $SP_G(s,t)$. □

Next we prove that $LABEL(.)$ can be used to correctly answer P2P distance queries. Then, we show how $label(.)$ possesses the essential information of $LABEL(.)$ for the processing of distance queries.

THEOREM 1. *Given a P2P distance query with two input vertices, s and t, let $\mathbb{X} = LABEL(s) \cap LABEL(t)$, then $dist_G(s,t) = \min_{w \in \mathbb{X}}\{dist_G(s,w) + dist_G(t,w)\}$ if $\mathbb{X} \neq \emptyset$, or $dist_G(s,t) = \infty$ if $\mathbb{X} = \emptyset$.*

PROOF. We first show that if $SP_G(s,t)$ exists, then $v_{max} \in \mathbb{X}$. Consider a sequence of vertices, $S = \langle s = u_1, u_2, \ldots, u_\alpha = v_{max} = v_\beta, \ldots, v_2, v_1 = t\rangle$, extracted from $SP_G(s,t)$, such that $\ell(u_1) < \ell(u_2) < ... < \ell(u_\alpha) = \ell(v_{max})$, $\ell(v_1) < \ell(v_2) < ... < \ell(v_\beta) = \ell(v_{max})$, and for $1 \leq i < \alpha$, any vertex w between u_i and u_{i+1} on $SP_G(s,t)$ has $\ell(w) < \ell(u_i)$, and same for any vertex between v_i and v_{i+1}. Note that since u_{i+1} is the next vertex after u_i with $\ell(u_{i+1}) > \ell(u_i)$, we have $\ell(w) \leq \ell(u_i)$, and $\ell(w) \neq \ell(u_i)$ by the vertex independence property.

Since u_i and u_{i+1} are connected, they must exist together in $G_{\ell(u_i)}$. Since there exists no other vertex w between u_i and u_{i+1} on $SP_G(s,t)$ such that $\ell(w) \geq \ell(u_i)$, u_i and u_{i+1} are not connected by any such w in $G_{\ell(u_i)}$. Thus, by Lemma 1, the edge (u_i, u_{i+1}) must exist in $G_{\ell(u_i)}$ for $G_{\ell(u_i)}$ to preserve the distance between u_i and u_{i+1}, which means that for $1 \leq j \leq \alpha$, u_j is an ancestor of s and hence $u_j \in LABEL(s)$. Note that $u_1 = s \in LABEL(s)$ if $\alpha = 1$. Similarly, we have $v_i \in LABEL(t)$, for $1 \leq i \leq \beta$. Thus, $v_{max} = u_\alpha = v_\beta \in \mathbb{X}$ and hence $dist_G(s,t) = dist_G(s, v_{max}) + dist_G(t, v_{max})$.

The other case is that $SP_G(s,t)$ does not exist, i.e., s and t are not connected, and we want to show that $\mathbb{X} = \emptyset$. Suppose on the contrary that there exists $w \in \mathbb{X}$. Then, it means that there is a path from s to w and from t to w, implying that s and t are connected, which is a contradiction. Thus, $\mathbb{X} = \emptyset$ and $dist_G(s,t) = \infty$ is correctly computed. □

Theorem 1 reveals two pieces of information that are essential for answering distance queries: the ancestor set and the distance to the ancestors maintained in $LABEL(.)$. We first show that $label(.)$ also encodes the same ancestor set of $LABEL(.)$.

LEMMA 4. *For each $v \in V_G$, $\mathcal{V}[label(v)] = \mathcal{V}[LABEL(v)]$.*

PROOF. First, we show that if $w \in \mathcal{V}[LABEL(v)]$, i.e., w is an ancestor of v, then $w \in \mathcal{V}[label(v)]$. According to the definition of ancestor, there exists a sequence $S = \langle v = w_1, w_2, ..., w_p = w\rangle$, such that $\ell(w_1) < \ell(w_2) < ... < \ell(w_p)$, and for $1 \leq i < p$, $(w_i, w_{i+1}) \in E_{G_{\ell(w_i)}}$. This definition implies that if w_i is currently in $\mathcal{V}[label(v)]$, w_{i+1} will also be added to $\mathcal{V}[label(v)]$ according to Definition 3. Since $w_1 = v$ must be in $\mathcal{V}[label(v)]$, it follows that $w = w_p$ is also in $\mathcal{V}[label(v)]$.

Next, we show that if $w \in \mathcal{V}[label(v)]$, then $w \in \mathcal{V}[LABEL(v)]$. First, we have $v \in \mathcal{V}[label(v)]$, v is also in $\mathcal{V}[LABEL(v)]$. Then, according to Definition 3, a vertex w is added to $\mathcal{V}[label(v)]$ only if $w \in adj_{G_{\ell(u)}}(u)$ for some u currently in $\mathcal{V}[label(v)]$, and $\ell(w) > \ell(u)$, and since u is an ancestor of v, it implies that w is an ancestor of v and hence $w \in \mathcal{V}[LABEL(v)]$. □

Next, we show that $label(.)$ also possesses the essential distance information for correct computation of P2P distance.

LEMMA 5. *Given a P2P distance query, s and t, let $\mathbb{X} = label(s) \cap label(t)$. If $SP_G(s,t)$ exists, then $v_{max} \in \mathbb{X}$, $d(s, v_{max}) = dist_G(s, v_{max})$ and $d(t, v_{max}) = dist_G(t, v_{max})$.*

PROOF. It follows from Lemma 4 that $label(s) \cap label(t) = LABEL(s) \cap LABEL(t)$. As the proof of Theorem 1 shows that $v_{max} \in LABEL(s) \cap LABEL(t)$, we also have $v_{max} \in \mathbb{X}$.

The proof of Theorem 1 defines a sequence, $S = \langle s = u_1, u_2, \ldots, u_\alpha = v_{max} = v_\beta, \ldots, v_2, v_1 = t\rangle$, extracted from

$SP_G(s,t)$. In particular, the proof shows that the edge (u_i, u_{i+1}) exists in $G_{\ell(u_i)}$ and $\ell(u_{i+1}) > \ell(u_i)$, for $1 \leq i < \alpha$. Thus, according to Definition 3, we add the entry $(u_{i+1}, (d(s,u_i) + \omega_{G_{\ell(u_i)}}(u_i, u_{i+1})))$ to $label(s)$. Since each $\omega_{G_{\ell(u_i)}}(u_i, u_{i+1})$ preserves the distance between u_i and u_{i+1}, and $d(s,u_1) = dist_G(s,u_1)$, it follows that $d(s, v_{max} = u_\alpha) = dist_G(s, v_{max} = u_\alpha)$. Similarly, we have $d(t, v_{max}) = dist_G(t, v_{max})$. □

Finally, the following theorem states the correctness of query processing using $label(.)$.

THEOREM 2. *Given a P2P distance query, s and t, $dist_G(s,t)$ evaluated by Equation 1 is correct.*

PROOF. The proof follows directly from Theorem 1, Lemmas 4 and 5. □

4. A K-LEVEL VERTEX HIERARCHY

In Definition 1, we do not limit the height h of the vertex hierarchy, i.e., the number of levels in the hierarchy. This definition ensures that an independent set L_i can always be obtained for each G_i, for $1 \leq i \leq h$. However, when the given graph is massive, there are two problems associated with the height of the vertex hierarchy. First, as the number of levels h increases, the label size of the vertices at the lower levels (i.e., vertices with a smaller level number) also increases. Since vertex labels require storage space and are directly related to query processing, there is a need to limit the vertex label size. Second, as we will discuss in Section 5, the complexity of constructing the vertex hierarchy is linear in h. Thus, reducing h can also improve the efficiency of index construction.

In this section, we propose to limit the height h by a k-level vertex hierarchy, where k is normally much smaller than h, and discuss how the above-mentioned problems are resolved.

4.1 Limiting the Height of Vertex Hierarchy

The main idea is to terminate the construction of the vertex hierarchy earlier at a level when certain condition is met. We first define the k-level vertex hierarchy.

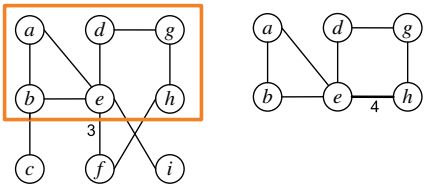
DEFINITION 4 (K-LEVEL VERTEX HIERARCHY). *Given a graph $G = (V_G, E_G, \omega_G)$, a vertex hierarchy structure $\mathbb{H} = (\mathbb{L}, \mathbb{G})$ of G, and an integer k, where $1 < k \leq (h+1)$ and h is the number of levels in $\mathbb{H}$, a **k-level vertex hierarchy** structure of G is defined by a pair $(\mathbb{H}_{<k}, G_k)$, where $\mathbb{H}_{<k}$ and G_k are defined as follows:*

- *$\mathbb{H}_{<k} = (\mathbb{L}_{<k}, \mathbb{G}_{<k})$ consists of the first $(k-1)$ levels of $\mathbb{H}$, i.e., $\mathbb{L}_{<k} = \{L_1, \ldots, L_{k-1}\}$ and $\mathbb{G}_{<k} = \{G_1, \ldots, G_{k-1}\}$;*
- *G_k is the same G_k as the G_k in $\mathbb{G}$.*

The k-level vertex hierarchy simply takes the first $(k-1)$ $L_i \in \mathbb{L}$, for $1 \leq i < k$, and the first k $G_i \in \mathbb{G}$, for $1 \leq i \leq k$. We set the value of k as follows: let i be the first level such that $(|G_i|/|G_{i-1}|) > \sigma$, where σ $(0 < \sigma \leq 1)$ is a threshold for the effect of G_i; then, $k = i$.

If $k = (h+1)$, then $\mathbb{H}_{<k}$ is simply $\mathbb{H}$ and G_k is an empty graph. In practice, a value of σ that attains a reasonable indexing cost and storage usage will often give $k \ll h$.

For the k-level vertex hierarchy, we assign the level number $\ell(v) = i$ for each vertex $v \in L(i)$, where $1 \leq i \leq (k-1)$, while for each vertex $v \in V_{G_k}$, we assign $\ell(v) = k$. In this way, we can compute $label(v)$ (or $LABEL(v)$) for each vertex $v \in V_G$ in the same way as discussed in Section 3.2. Note that $label(v) = \{(v,0)\}$ for each vertex $v \in V_{G_k}$ since v has the highest level number among all vertices in V_G.



(a) $G = G_1$, L_1={c, f, i} (b) G_2

Figure 3: A k-level vertex hierarchy ($k = 2$)

EXAMPLE 5. *Let us consider our running example in Figure 1, if we set $k = 2$, there is only one level L_1 in $\mathbb{L}_{<k}$, the graph G_2 is the highest level graph and is not further decomposed. The k-level vertex hierarchy is shown in Figure 3. The maximum level of vertices is 2, since all vertices v in G_2 are assigned $\ell(v) = 2$. The labels for the vertices in L_1 are shown in the following table.*

$label(c)$	$\{(b,1),(c,0)\}$
$label(f)$	$\{(e,3),(f,0),(h,1)\}$
$label(i)$	$\{(e,1),(i,0)\}$

4.2 P2P Distance Querying by k-Level Vertex Hierarchy

According to Section 4.1, $\ell(v)$ and $label(v)$ computed from the k-level vertex hierarchy may be different from those computed from the original vertex hierarchy. However, we show later in this section that these labels are highly useful for they capture all the information that is essential from $G - G_k$ for a continued distance search in G_k. Given a P2P distance query, s and t, we process the query according to whether s and t are in G_k. We have the following two possible types of queries.

Type 1: $s \notin V_{G_k}$ and $t \notin V_{G_k}$, and either $(\mathcal{V}[label(s)] \cap V_{G_k}) = \emptyset$ or $(\mathcal{V}[label(t)] \cap V_{G_k}) = \emptyset$. Type 1 queries are evaluated by Equation 1.

Type 2: queries that are not Type 1. Type 2 queries are evaluated by a *label-based bi-Dijkstra search* procedure.

We have discussed query processing by Equation 1 in Section 3.3. We now discuss how we process Type 2 queries as follows.

4.2.1 Label-based bi-Dijkstra Search

We describe a bidirectional Dijkstra's algorithm that utilizes vertex labels for effective pruning. The algorithm consists of two main stages: (1) initialization of distance queues and pruning condition, and (2) bidirectional Dijkstra search.

As shown in Algorithm 1, we first initialize a *forward* and a *reverse* min-priority queue, *FQ* and *RQ*, which are to be used for running Dijkstra's single-source shortest path algorithm from s and t, respectively. For any vertex $v \in V_{G_k}$, if $(v, d(s,v)) \in label(s)$, we add $(v, d(s,v))$ to *FQ* with $d(s,v)$ as the key. For all other vertices in V_{G_k} but not in $label(s)$, we add the record (v, ∞) to *FQ*. Similarly, we initialize *RQ*.

The vertex labels can also be used for pruning the search space. If there exists a path between s and t that passes through some vertex $w \in (V_G - V_{G_k} - \{s,t\})$, then Lines 5-6 initializes μ as the minimum length of such a path. Note that $\mu \geq dist_G(s,t)$.

We now describe Stage 2 of the query processing. We run Dijkstra's algorithm simultaneously from s and t by extracting the vertex v with the minimum key from *FQ* or *RQ* (Line 9). Let $(v, d(x,v))$ be the extracted record, where $x = s$ if the record

Algorithm 1: Label-based bi-Dijkstra Search

Input : s, t, $label(s)$, $label(t)$, G_k
Output : $dist_G(s,t)$
// Stage 1: initialization of distance queues and pruning condition
// FQ (RQ): forward (reverse) min-priority queue
1 initialize *FQ* with the set $\{(v, d(s,v)) : v \in V_{G_k}, (v, d(s,v)) \in label(s)\}$, with $d(s,v)$ as the key;
2 initialize *RQ* with the set $\{(v, d(t,v)) : v \in V_{G_k}, (v, d(t,v)) \in label(t)\}$, with $d(t,v)$ as the key;
3 $\forall\, v \in V_G$ and v not in *FQ*(*RQ*), insert (v, ∞) into *FQ*(*RQ*);
// μ: shortest distance from s to t found so far
// μ is used for pruning in Stage 2
4 $\mu \leftarrow \infty$;
5 $\mathbb{X} \leftarrow label(s) \cap label(t)$;
6 **if** $\mathbb{X} \neq \emptyset$ **then** $\mu \leftarrow \min_{w \in \mathbb{X}}\{d(s,w) + d(w,t)\}$;

// Stage 2: bidirectional Dijkstra search
7 $S \leftarrow \emptyset$;
8 **while** *both FQ and RQ are not empty, and* $(\min(FQ) + \min(RQ)) < \mu$ **do**
9 $(v, d(x,v)) \leftarrow$ *extract-min*(FQ, RQ) ; // $x = s$ or $x = t$
10 let $x' = t$ if $x = s$, and $x' = s$ if $x = t$;
11 **if** $\langle v, dist_G(x,v)\rangle$ *is not in* S **then**
12 insert $\langle v, dist_G(x,v)\rangle$ into S;
13 **foreach** $u \in adj_{G_k}(v)$ **do**
14 **if** $d(x,u) > d(x,v) + \omega_{G_k}(v,u)$ **then**
15 $d(x,u) \leftarrow d(x,v) + \omega_{G_k}(v,u)$;
16 update $d(x,u)$ in *FQ* (if $x = s$) or *RQ* (if $x = t$);
17 **if** $\langle u, dist_G(x',u)\rangle$ *is in* S **then**
18 $\mu \leftarrow \min\{\mu, d(x,u) + dist_G(x',u)\}$;
19 **return** μ;

is extracted from *FQ* and $x = t$ otherwise. At this point, Dijkstra's algorithm guarantees that the distance from x to v is found, i.e., $d(x,v) = dist_G(x,v)$. Then, in Lines 13-18, the distance from x to every neighbor u of v in G_k is updated, if u is still in *FQ* (if $x = s$) or *RQ* (if $x = t$).

In addition to starting the search in both directions from s and t in Dijkstra's algorithm, we also add a pruning condition in Line 8 that requires the sum of the minimum keys of *FQ* and *RQ* to be less than μ. If this sum is not less than μ, then it means that no path from s to t of a shorter distance than μ can be found (proved in Theorem 4) and hence we return $dist_G(s,t) = \mu$.

To improve the pruning effect so as to converge the search quickly, we keep updating μ whenever $d(x,u)$ is updated if $dist_G(x',u)$ has been found (Lines 17-18), since u is a potential vertex on $SP_G(s,t)$. We use a set S to keep a set of vertices whose distance from s or t has been found. Whenever $dist_G(x,v)$ is found for a vertex v, if v is not yet in S, we insert v, together with $dist_G(x,v)$, into S.

We give an example to illustrate how queries are processed as follows.

EXAMPLE 6. *Let us consider Example 5. Suppose we need to process a distance query between vertices c and i, i.e. $s = c$, $t = i$. In $label(c)$, b is in G_k, and therefore we enter $(b, d(c,b) = 1)$ into FQ. In $label(i)$, e is in G_k, hence we enter $(e, d(i,e) = 1)$ into RQ. $label(c) \cap label(i) = \phi$, hence $\mu = \infty$ after Stage 1 of Algorithm 1. In Stage 2, let us extract $(b,1)$ from FQ first, $\langle b, 1\rangle$ is inserted into S, and we enter $(a,2)$, $(e,2)$, into FQ. Next we extract $(e,1)$ from RQ, and insert $\langle e,1\rangle$ into S. $(a,2)$, $(d,2)$, $(b,2)$ are entered into RQ. Since b is in S, we update μ to $2+1$ = 3. At this point $(min(FQ) + min(RQ)) > \mu$ and we return $dist_G(c,i) = 3$.*

4.2.2 Correctness

We now prove the correctness of query processing by the k-level vertex hierarchy. We first prove the correctness for processing Type 1 queries.

THEOREM 3. *Given a P2P distance query, s and t, if the query belongs to Type 1, then $dist_G(s,t)$ evaluated by Equation 1 is correct.*

PROOF. First, we show that if the query belongs to Type 1, then $SP_G(s,t)$ does not contain any vertex in V_{G_k}. Suppose on the contrary that $SP_G(s,t)$ contains a vertex in V_{G_k}. Then, consider the sub-path of $SP_G(s,t)$ from s to x, where x is the only vertex on the sub-path that is in V_{G_k}. Since $SP_G(s,t)$ is a shortest path in G, this sub-path is a shortest path from s to x in G. Let $SP_G(s,x)$ be the sub-path. Consider the query with two input vertices s and x; then, by similar argument as in the proof of Lemma 3 we have $v_{max} = x$ on $SP_G(s,x)$, and by similar argument as in the proof of Lemma 5 we have $x = v_{max} \in \mathcal{V}[label(s)]$. A symmetric analysis on the sub-path from t to some vertex y, where y is the only vertex on the sub-path that is in V_{G_k}, shows that $y = v_{max}$ on $SP_G(t,y)$ and $y \in \mathcal{V}[label(t)]$. This contradicts the definition of Type 1 query that either $(\mathcal{V}[label(s)] \cap V_{G_k}) = \emptyset$ or $(\mathcal{V}[label(t)] \cap V_{G_k}) = \emptyset$.

Now if $SP_G(s,t)$ does not contain any vertex in V_{G_k}, then the query can be answered using only label entries of vertices from the first $(k-1)$ levels of the vertex hierarchy. These entries will have identical occurrences and contents in the vertex labels at the first k levels of any vertex hierarchy $\mathbb{H}_{<j}$, where $k \leq j \leq h+1$, which is formed by limiting the height of a given $\mathbb{H}$. Thus, the correctness of query answer follows from Theorem 2. □

Note that Type 1 queries exist only if there exist more than one connected component in G such that all vertices in some connected component(s) have a level number lower than k.

Next we prove the correctness for processing Type 2 queries.

THEOREM 4. *Given a P2P distance query, s and t, if the query belongs to Type 2, then $dist_G(s,t)$ evaluated by the label-based bi-Dijkstra search procedure is correct.*

PROOF. We have two cases: (1) $SP_G(s,t)$ does not contain any vertex in V_{G_k}, or (2) otherwise.

If $SP_G(s,t)$ does not contain any vertex in V_{G_k}, then $dist_G(s,t)$ is computed in Lines 5-6 of Algorithm 1, or in other words by Equation 1. As explained in the proof of Theorem 3, the correctness of query answer follows from Theorem 2.

If $SP_G(s,t)$ contains at least one vertex in V_{G_k}, then consider the two subpaths, $SP_G(s,x)$ and $SP_G(t,y)$, defined in the proof of Theorem 3 (note that it is possible $s = x$ and/or $x = y$ and/or $y = t$). $dist_G(s,x)$ and $dist_G(t,y)$ can be answered using only label entries of vertices in $\mathbb{L}_{<k}$ and their ancestors in G_k for $(\mathbb{H}_{<k}, G_k)$. From the labeling mechanism, the occurrences and contents of such label entries will be identical in the labels of vertices in the first k levels of any vertex hierarchy $\mathbb{H}_{<j}$, $k \leq j \leq h+1$, which is formed by limiting the height of a given $\mathbb{H}$. Hence by Theorem 2, $dist_G(s,x)$ and $dist_G(t,y)$ are correctly initialized in Lines 1-3 of Algorithm 1. Thus, if we do not consider the pruning condition in Line 8, then Dijkstra's algorithm guarantees the distance from s (and t) to any vertex in G_k correctly computed, from which we can obtain $dist_G(s,t)$.

Now we consider query processing with pruning. Let $\mu = \mu*$, and $min_f = \min(FQ)$ and $min_r = \min(RQ)$, when the search

stops. If $\mu*$ is the value of μ initialized in Line 6, then we must have $x = y \in (label(s) \cap label(t))$ and hence $\mu* = (dist_G(s,x) + dist_G(t,x))$. Otherwise, $\mu*$ is a value assigned to μ in Line 18 and suppose to the contrary that there exists a shorter path between s and t with length p such that $p < \mu*$. Since the path passes through vertices in G_k, there must exist an edge (v,u) in G_k such that $p = dist_G(s,v) + \omega_{G_k}(v,u) + dist_G(u,t)$, $dist_G(s,v) < min_f$ and $dist_G(u,t) < min_r$. The existence of this edge is guaranteed because $p < \mu* \leq (min_f + min_r)$. Since $dist_G(s,v) < min_f$ and $dist_G(u,t) < min_r$, by Dijkstra's algorithm, both $dist_G(s,v)$ and $dist_G(t,u)$ have been computed when the search stops. Thus, μ should have been updated to a value not greater than p in Line 18 when the edge (v,u) was processed. This contradicts our assumption and hence $\mu* = dist_G(s,t)$. □

4.3 Handling Directed Graphs

To handle directed graphs, we need to make the following changes. Let us use (u,v) to indicate a directed edge from u to v in this subsection. The concept of independent set can be applied in the same way by simply ignoring the direction of the edges. However, for distance preservation, we create an augmenting edge (u,w) at G_i only if $\exists v \in L_{i-1}$ such that $(u,v),(v,w) \in E_{G_{i-1}}$. We distinguish two types of ancestors for a vertex v: **in-ancestors** and **out-ancestors**. The definition of in-ancestors is similar to that of ancestors in undirected graphs, except that we only consider edges from higher-level vertices to lower-level vertices. Analogously, the definition of out-ancestors concerns edges going from lower-level vertices to higher-level vertices.

The labeling needs to handle two directions. For each vertex v, we need two types of labels defined as follows. The **in-label** of a vertex $v \in V_G$, denoted by $LABEL_{in}(v)$, is defined as $LABEL_{in}(v) = \{(u, dist_G(u,v)) : u \in V_G \text{ is an in-ancestor of } v\}$. The **out-label** of a vertex $v \in V_G$, denoted by $LABEL_{out}(v)$, is defined as $LABEL_{out}(v) = \{(u, dist_G(v,u)) : u \in V_G \text{ is an out-ancestor of } v\}$.

Given a P2P distance query with two input vertices, s and t, we compute $\mathbb{X} = LABEL_{out}(s) \cap LABEL_{in}(t)$ and then answer the query in the same way as given in Equation 1.

5. ALGORITHMS

In this section, we present the algorithms for index construction (i.e., vertex hierarchy construction and vertex labeling) and query processing using the vertex labels. In recent years, due to the proliferation of many massive real world networks, there has been an increasing interest in algorithms that handle large graphs. For processing large graphs that cannot fit in main memory, I/O cost usually dominates. Thus, we propose I/O-efficient algorithms, from which the in-memory algorithms can also be easily devised.

For the analysis of the I/O complexity in this section, we define the following notation [5]. Let $scan(N) = \Theta(N/B)$ and $sort(N) = \Theta(\frac{N}{B} \log_{M/B} \frac{N}{B})$, where N is the amount of data being read or written from/to disk, M is the main memory size, and B is the disk block size ($1 \ll B \leq M/2$).

5.1 Algorithm for Index Construction

Although the vertex hierarchy, except G_k, is not required for query processing, it is needed for vertex labeling. The vertex hierarchy consists of two components, $\mathbb{L}$ and $\mathbb{G}$; thus we have two main steps: (1) computing each independent vertex set $L_i \in \mathbb{L}$, and (2) constructing each distance-preserving graph $G_i \in \mathbb{G}$. We first describe these two steps, followed by the construction of the overall vertex hierarchy, and finally the vertex labeling.

Algorithm 2: Constructing L_i

Input : A graph $G_i = (V_{G_i}, E_{G_i}, \omega_{G_i})$
Output : L_i and $ADJ(L_i) = \{adj_{G_i}(v) : v \in L_i\}$

1 allocate a buffer for L_i and $ADJ(L_i)$, and a buffer for L';
2 $G'_i \leftarrow G_i$;
3 sort $adj_{G'_i}(v)$ in G'_i in ascending order of $deg_{G'_i}(v)$;
4 **foreach** $adj_{G'_i}(u)$ *read in* G'_i **do**
5 **if** $u \notin L'$ **then**
6 insert u into L_i, and insert $adj_{G'_i}(u)$ into $ADJ(L_i)$;
7 **foreach** $v \in adj_{G'_i}(u)$ **do**
8 **if** $v \notin L'$ **then** insert v into L';
9 **if** *buffer for* L_i *and* $ADJ(L_i)$ *is full* **then** flush the buffer;
10 **if** *buffer for* L' *is full* **then**
11 scan G'_i to delete all $v \in L'$ and $adj_{G'_i}(v)$, and clear L';

Algorithm 3: Constructing G_i

Input : G_{i-1}, L_{i-1} and $ADJ(L_{i-1})$
Output : G_i

1 $G_i \leftarrow G_{i-1}$;
2 remove from G_i all $v \in L_{i-1}$ and $adj_{G_{i-1}}(v)$;
3 $E_A \leftarrow \emptyset$;
4 **foreach** $adj_{G_{i-1}}(v) \in ADJ(L_{i-1})$ **do**
5 **foreach** $u, w \in adj_{G_{i-1}}(v)$, *where* $u < w$ **do**
6 insert into E_A the edges (u,w) and (w,u), with $\omega_{G_i}(u,w) = \omega_{G_i}(w,u) = (\omega_{G_{i-1}}(u,v) + \omega_{G_{i-1}}(v,w))$;
7 sort the edges in E_A by vertex ID's;
8 scan E_A and G_i to add each edge $(u,w) \in E_A$ to G_i, or update $\omega_{G_i}(u,w)$ with the smaller weight if (u,w) already exists in G_i;

5.1.1 Constructing L_i

We want to maximize the size of each L_i as this helps to minimize the number of levels h and hence also minimizes the vertex label size. However, maximizing L_i means computing the maximum independent set of G_i, which is an NP-hard problem.

We adopt a greedy strategy to approximate the set of maximum independent set of G_i by selecting the vertex with minimum degree at each step [20], since small degree vertices have smaller number of dependent (i.e., adjacent) vertices and hence more vertices are left as candidates for independent set at the next step. Moreover, the greedy algorithm can also be easily extended to give an I/O-efficient algorithm that handles the case when G_i is too large to fit in main memory, as described in Algorithm 2.

The algorithm computes an independent set L_i of G_i, together with the adjacency lists of the vertices in L_i, denoted by $ADJ(L_i)$. We use $ADJ(L_i)$ to construct G_{i+1} in Section 5.1.2. To compute L_i, we also keep those vertices that have been excluded from L_i in the algorithm, as denoted by L'. We use a buffer to keep the current L_i and $ADJ(L_i)$, and another buffer to keep L'.

The algorithm first makes a copy of G_i, let it be G'_i, and then sorts the adjacency lists in G'_i in ascending order of the vertex degrees (i.e., the sizes of the adjacency lists). Then, we read G'_i in this sorted order, i.e., the adjacency lists of vertices with smaller degrees are read first. For each $adj_{G'_i}(u)$ read, if u is not in L', we include u into L_i and add $adj_{G'_i}(u)$ to $ADJ(L_i)$. Meanwhile, we exclude all vertices in $adj_{G'_i}(u)$ from L_i because of their dependence with u, i.e., we add these vertices to L'. The algorithm terminates when $adj_{G'_i}(u)$ for all u in G'_i are read.

If G_i is very large, it is possible that L_i and $ADJ(L_i)$ are too

large to be kept by a memory buffer. We can simply write the current L_i and $ADJ(L_i)$ in the buffer to disk, and then clear the buffer for new contents of L_i and $ADJ(L_i)$. However, when the buffer for L' is full, we cannot simply flush the buffer since it is possible that $\exists u \in L'$, $adj_{G'_i}(u)$ has not been read yet. To tackle this without incurring random disk accesses, we scan G'_i to remove all the vertices currently in L', together with their adjacency lists, from G'_i, because these vertices have already been excluded from L_i. Then, we clear the buffer for L'.

If G'_i can be resident in main memory, Lines 10-11 of Algorithm 2 are not necessary and we only need to scan G'_i once. If G'_i is resident on disk, it is easy to see that only sequential scans of G'_i are needed and expensive random disk access is avoided.

Algorithm 2 takes $sort(|G_i|)$ I/Os to sort G_i. If $|L'| < M$, we need another $scan(|G_i|)$ I/Os to read G_i. Otherwise, $O(|L'|/M) * scan(|G_i|)$ I/Os are required.

5.1.2 Constructing G_i

After obtaining L_{i-1} and $ADJ(L_{i-1})$, we use them to construct G_i. As shown in Algorithm 3, we first initialize G_i by removing the occurrences of all vertices in L_{i-1}, together with their adjacency lists, from G_{i-1}. However, the resultant G_i may not satisfy the distance preservation property. As discussed in Section 3.1, the violation to this property can be fixed by the creation of a set of augmenting edges. We create these augmenting edges from $ADJ(L_{i-1})$ as follows.

When a vertex $v \in L_{i-1}$, together with $adj_{G_{i-1}}(v)$, is removed from G_{i-1} to form G_i, what is missing in G_i is the path $\langle u, v, w\rangle$ for any $u, w \in adj_{G_{i-1}}(v)$, where $u < w$ (i.e., u is ordered before w). Thus, to preserve the distance we only need to create the augmenting edge (u, w), and symmetrically (w, u) for undirected graphs, with weight $(\omega_{G_{i-1}}(u, v) + \omega_{G_{i-1}}(v, w))$.

We create all such augmenting edges in Lines 4-6 of Algorithm 3 and store them in an array E_A. Then, we sort the edges in E_A first in ascending order of the first vertex and then of the second vertex. Then, we scan both E_A and G_i (already sorted in its adjacency list representation), so that each edge in E_A is merged into G_i. If an edge in E_A is already in G_i, then its weight is updated to the smaller value of its weight recorded in E_A and in G_i.

If main memory is not sufficient, Line 2 of Algorithm 3 uses $O(|L_{i-1}|/M) * scan(|G_{i-1}|)$ I/Os, if $|E_A| < |G_i|$ then Lines 3-6 and 8 use $scan(|G_i|)$ I/Os, and Line 7 uses $sort(|G_i|)$ I/Os, else we may sort partial E_A each time it fills up memory and then scan G_i, so the IO cost is $O(|E_A|/M)scan(|G_i|)$.

5.1.3 Constructing $(\mathbb{L}, \mathbb{G})$

The overall scheme to construct the vertex hierarchy, $(\mathbb{L}, \mathbb{G})$, is to start with the given $G_1 = G$, and keep repeating the two steps of computing L_i (Algorithm 2) and constructing G_i (Algorithm 3) until we reach a level k (see Section 4.1 for the value of k).

5.1.4 Top-Down Vertex Labeling

Definition 3 essentially defines a procedure for computing $label(v)$ for each $v \in V_G$. However, a careful analysis will show that such a procedure, if implemented directly as it is described, involves much redundant processing as implied by the following corollary of Lemma 4.

COROLLARY 1. *Given a vertex $v \in L_i$, we have $\mathcal{V}[label(v)] = \{v\} \cup (\bigcup_{u \in adj_{G_i}(v)} \mathcal{V}[label(u)])$.*

PROOF. By Definition 3, $\forall u \in adj_{G_i}(v)$, u will be included into $\mathcal{V}[label(v)]$. From the result of Lemma 4, we have $\forall u \in \mathcal{V}[label(v)]$, u is an ancestor of v by Definition 2. In the same way, we have $\forall w \in \mathcal{V}[label(u)]$, $w \in \mathcal{V}[label(v)]$ since w is then also an ancestor of v. Thus, $\forall u \in adj_{G_i}(v)$, $\mathcal{V}[label(u)] \subseteq \mathcal{V}[label(v)]$.

Next, $\forall w \in \mathcal{V}[label(v)] \backslash \{v\}$, $w \in \mathcal{V}[label(u)]$ for some $u \in adj_{G_i}(v)$ because w is included into $\mathcal{V}[label(v)]$ from some u by Definition 3, and by the same procedure w will be included into $\mathcal{V}[label(u)]$ when we compute $label(u)$. □

Algorithm 4: Top-Down Vertex Labeling

```
   Input  : (𝕃, 𝔾)
   Output : label(v), ∀v ∈ V_G
   // Initialization of vertex labels
1  for i = 1, ..., k − 1 do
2      foreach v ∈ L_i do
3          label(v) ← {(v, 0)} ∪ {(u, ω_{G_i}(v, u)) : u ∈ adj_{G_i}(v)};
4  ∀v ∈ V_{G_k}:  label(v) ← {(v, 0)};

   // Top-down vertex labeling
5  for i = k − 1, ..., 1 do
6      allocate buffer B_L and load label(v), for each v ∈ L_i, in B_L;
7      allocate buffer B_U and load label(v), for each v ∈ L_j for
       i < j < k and for each v ∈ V_{G_k}, in B_U;
8      foreach block B_L do
9          foreach block B_U do
10             foreach label(v) in B_L do
11                 foreach label(u) in B_U do
12                     if (u, d(v, u)) ∈ label(v) then
13                         foreach (w, d(u, w)) ∈ label(u) do
14                             if (w, d(v, w)) ∉ label(v) then
15                                 add (w, d(v, u) + d(u, w)) to
                                   label(v);
16                             else
17                                 d(v, w) =
                                   min(d(v, w), d(v, u)+d(u, w));
```

Corollary 1 implies that $label(v)$ can be computed from $label(u)$, for each $u \in adj_{G_i}(v)$, instead of from scratch. Based on this, we design a more efficient top-down algorithm for vertex labeling as shown in Algorithm 4.

The algorithm consists of two stages: initialization of vertex labels and top-down vertex labeling by block nested loop join, discussed as follows.

According to Corollary 1, we only need to add $(v, 0)$ and $(u, \omega_{G_i}(v, u))$ for all $u \in adj_{G_i}(v)$ to $label(v)$, and then derive other entries of $label(v)$ from $label(u)$ in the top-down process. For each $v \in V_{G_k}$, however, we only need to add $(v, 0)$ to $label(v)$ since each $v \in V_{G_k}$ has only one ancestor, i.e., v itself.

After the initialization, we compute the labels for the vertices starting from the top levels to the bottom levels, i.e., from level $(k-1)$ down to level 1. We assume that the set of labels at each level may not be able to fit in main memory and hence use block nested loop join to find the matching labels, i.e., $label(u)$ for each $u \in adj_{G_i}(v)$ when we process v at level i. Note that if $u \in adj_{G_i}(v)$, then $(u, d(v, u)) \in label(v)$ by the initialization. Thus, as shown in Lines 11-16, we derive the entries of other ancestors of v from $label(u)$ directly, which essentially follows the rule specified in Definition 3.

The complexity of the algorithm is apparently dominated by the top-down process. Let $b_L(i) = |\{label(v) : v \in L_i\}|$, and $b_U(i) = |\bigcup_{i<j<k}\{label(v) : v \in L_j\} \cup \{label(v) : v \in V_{G_k}\}|$.

The I/O complexity for the block nested loop join is given by $(b_L(i)/M) * (b_U(i)/B)$. Thus, the I/O complexity of Algorithm 4 is given by $O(\sum_{i=1}^{k-1}((b_L(i)/M) * (b_U(i)/B)))$.

5.2 Algorithm for Query Processing

For processing large datasets, the vertex labels may not fit in main memory and are stored on disk. The entries in each $label(v)$ are stored sequentially on disk and are sorted by the vertex ID's of the ancestors of v. Thus, $label(s) \cap label(t)$ involves simple sequential scanning of the entries in $label(s)$ and $label(t)$. From our experiments, the vertex labels are small in size and retrieving a vertex label from disk takes only one I/O. The CPU time for query processing comes mostly from the bi-Dijkstra search. For a graph $G = (V, E)$, a binary heap can be used and Dijkstra's algorithms runs in $O((|E| + |V|) \log |V|)$ time.

6. EXPERIMENTAL EVALUATION

We evaluate the performance of our method and compare with other related methods for processing P2P distance queries [22, 38]. All methods tested were programmed in C++ and compiled with the same compiler. All experiments were run on a computer with an Intel 3.3 GHz CPU, 4GB RAM and a 7200 RPM SATA hard disk, running Ubuntu 11.04 Linux OS.

We use both undirected and directed real datasets. For the undirected datasets: BTC is a semantic graph converted from the Billion Triple Challenge 2009 RDF dataset (vmlion25.deri.ie/), where each vertex represents an object such as a person, a document, and an event, and each edge represents the relationship between two nodes such as "has-author", "links-to", and "has-title". Web (barcelona.research.yahoo.net/webspam) is a subgraph of the UK Web graph, where vertices are pages and edges are hyperlinks. The original graph $\vec{G}$ is directed and converted into undirected graph G as follows: if two vertices are reachable from each other within w hops in $\vec{G}$, where $w \in \{1, 2\}$, they have an undirected edge with weight w in G. Since there are many connected components in G, we extract the largest connected component . As-Skitter is an Internet topology graph from traceroutes run daily in 2005 (www.caida.org/tools/measurement/skitter). Email-Enron is the communication network from Enron. Ca-Astroph is the collaboration network of Arxiv Astro Physics. The homo and erdos datasets are a biological network and a social network from [38]. For the directed graphs: ukweb is a subgraph of the UK Web graph. The wiki-talk graph is a communication network from Wikipedia. Email-EuAll is the email network from a EU research institution. Soc-sign-slashdot and soc-Epinions are social networks. The p2p graph is the Gnutella peer to peer network. Cit-HepPh is the Arxiv High Energy Physics paper citation network. The wiki-vote graph is the Wikipedia who-votes-on-whom network. More details of the following datasets can be found in (snap.stanford.edu): as-Skitter, email-Enron, aa-Astroph, wiki-talk, email-EuAll, soc-sign-slashdot, soc-Epinions, p2p, cit-HepPh, and wiki-vote. We list the datasets in Table 1.

	$\|V\|$	$\|E\|$	Avg. deg	Max. deg	Disk size
Undirected graphs					
BTC	164.7M	361.1M	2.19	105,618	5.6 GB
Web	6.9M	113.0M	16.40	31,734	1.1 GB
as-Skitter	1.7M	22.2M	13.08	35,455	200 MB
Email-Enron	37K	368K	10.02	1,383	2.7MB
ca-Astroph	19K	396K	21.10	504	2.8MB
homo	7K	40K	5.64	157	0.3MB
erdos	7K	24K	3.42	507	0.2MB
Directed graphs					
ukweb	105.9M	297.4M	2.81	78,228	7.4GB
wiki-talk	2.4M	5.0M	2.10	100,022	104.2MB
Email-EuAll	265K	419K	1.58	7,631	8.6MB
soc-sign-slashdot	77K	517K	6.68	2,532	8MB
soc-Epinions	76K	509K	6.71	3,035	7.5MB
p2p	63K	148K	2.36	78	2.8MB
cit-HepPh	35K	421K	12.20	846	6.3MB
wiki-vote	7K	104K	14.57	893	1.4MB

Table 1: Real datasets

6.1 Comparison with Other Methods

We compare with two most recent works on point-to-point distance querying, **HCL** [22] for directed graphs and **TEDI** [38] for undirected graphs. In [22, 38], HCL and TEDI were shown to outperform other existing methods for distance querying in directed and undirected graphs, respectively. Since IS-Label uses bidirectional Dijkstra search (**BDIJ**) in G_k, we also report the results of BDIJ on the original graph G as a reference. We use $\sigma = 0.95$ as our default threshold in IS-Label to obtain G_k, i.e., when $(|G_i|/|G_{i-1}|) > \sigma$, we set $k = i$. To assess query performance, we randomly generate 1000 queries in each dataset and compute the average query time.

Table 2 reports both the indexing performance and query performance of all the methods. The results clearly show that IS-Label significantly outperforms both HCL and TEDI in all aspects. First, IS-Label is far more scalable than HCL and TEDI. IS-Label can process graphs up to three and four orders of magnitude larger than those HCL and TEDI can handle. For most of the graphs tested, we were not able to obtain the result for HCL and TEDI due to the prohibitively high cost of all-pairs shortest path computation. Second, even for processing those small graphs which HCL and TEDI can handle, both the indexing time and querying time of IS-Label are up to orders of magnitude shorter than those of HCL in all cases. TEDI is faster in indexing than IS-Label but TEDI is only able to handle very small graphs with only 7K vertices.

For the relatively smaller graphs, IS-Label obtains a complete vertex hierarchy, i.e., $k = (h+1)$ and G_k is an empty graph, but the indexing process is still very fast. The label size is in general larger than when we have a non-empty G_k, but the largest overall label size is only around 1 GB which is certainly acceptable. Note that if storage space is critical, we can easily limit the height of the vertex hierarchy to obtain a smaller label size, as we have done for the large graphs. This shows the flexibility of our labeling scheme for processing graphs of different sizes and/or computing resources.

Compared with BDIJ, IS-Label is significantly faster for processing all datasets. The need for an index becomes clearer for processing larger graphs. For processing the datasets BTC and ukweb, we could not obtain the result for BDIJ since the memory consumption of BDIJ exceeds the main memory capacity.

In conclusion, our labeling scheme is a big step forward in the development of an efficient index for answering point-to-point distance queries in real world general graphs (both directed and undirected). This is evident from the results that IS-Label is able to handle graphs up to three and four orders of magnitude larger than the largest graphs used to test HCL [22] and TEDI [38], respectively. Furthermore, even for processing small graphs, IS-Label also significantly outperforms the most recent work, HCL [22], in all aspects.

6.2 Effect of Graph Density

In this experiment, we examine the effect of density on the performance of IS-Label. We generate two types of synthetic graphs using the graph generator provided by [39]: the *preferential attachment* model (modeling graphs with power-law degree distribution) [10], and the *small world* model (modeling graphs with short aver-

	k	$\|V_{G_k}\|$	$\|E_{G_k}\|$	$\|label\|$	Index size			Indexing time (s)			Query time (ms)			
Data graphs					IS-Label	HCL	TEDI	IS-Label	HCL	TEDI	IS-Label	HCL	TEDI	BDIJ
Undirected														
BTC	6	134K	16.4M	6.6	7.1GB	–	–	2057.98	–	–	6.35	–	–	–
Web	19	242K	14.5M	259.1	8.1GB	–	–	2034.07	–	–	28.39	–	–	108.53
as-Skitter	6	86K	8.5M	53.2	428.6MB	–	–	487.92	–	–	2.32	–	–	4.91
Email-Enron	78	0	0	784.1	137.7MB	46.4MB	–	36.58	51780	–	0.005	0.294	–	0.107
ca-Astroph	164	0	0	2605.7	233.5MB	78.4MB	–	238.56	22445	–	0.015	1.818	–	0.091
homo	75	0	0	466.7	15.7MB	9.4MB	27.9MB	16.70	2871	3.12	0.003	0.223	0.0014	0.024
erdos	17	0	0	27.7	1MB	2.2MB	1.9MB	2.93	1889	0.22	0.0008	0.012	0.0009	0.01
Directed														
ukweb	10	1.1M	54.5M	7.0	8.9GB	–	n.a.	10132.8	–	n.a.	19.796	–	n.a.	–
wiki-talk	4	14K	1.1M	1.5	85MB	–	n.a.	39.93	–	n.a.	0.011	–	n.a.	0.198
Email-EuAll	4	975	125K	1.0	8.0MB	–	n.a.	1.39	–	n.a.	0.008	–	n.a.	0.021
soc-sign-slashdot	215	0	0	1336.5	1GB	–	n.a.	439.47	–	n.a.	0.007	–	n.a.	0.048
soc-Epinions	216	0	0	1586.3	1.1GB	–	n.a.	517.41	–	n.a.	0.009	–	n.a.	0.079
p2p	322	0	0	896.3	536.3MB	–	n.a.	464.82	–	n.a.	0.004	–	n.a.	0.045
cit-HepPh	107	0	0	325	107.8MB	63.9MB	n.a.	44.41	40747	n.a.	0.002	0.246	n.a.	0.445
wiki-vote	69	0	0	176.2	12.1MB	2.6MB	n.a.	13.20	559	n.a.	0.001	0.018	n.a.	0.005

Table 2: Performance results of IS-Label, HCL, TEDI, and Bi-Dijkstra (BDIJ): the size of G_k and average number of entries per label, $|label|$, of IS-Label; and the total index size, total indexing time, and average query time of all methods.

age distance and small communities) [37]. We fix the graph size to 1 million vertices and vary the average degree of the vertices from 5 to 80.

We report the total label size for each average degree value in Figure 4. The indexing time and query time follow similar trend as the label size.

The results show that our label size is not significantly increased when the density increases for both preferential attachment and small-world models. The more rapid increase in the label size when the degree increases from 5 to 20 is because a larger k, i.e., more levels in the vertex hierarchy, is computed for smaller graphs. However, when the density becomes larger, the increase in the label size becomes small.

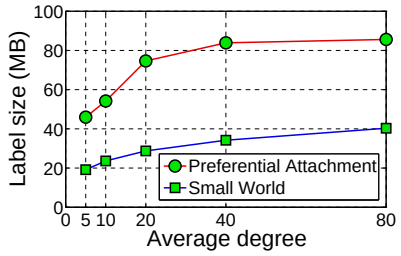


Figure 4: Performance results of density change

6.3 Effect of k-Level Vertex Hierarchy

For handling large graphs, IS-Label restricts the number of levels in the vertex hierarchy to k in order to combat the increasing label size, and then run the bi-Dijsktra algorithm on G_k. In this experiment, we test the effect of k on the performance. In particular, we examine whether G_k is too dense for the bi-Dijsktra algorithm to be efficient.

We report the results in Table 3, where we use the two larger graphs BTC and Web to demonstrate the effect of k (the results on other datasets are similar). The value of k computed by our algorithm is 6 and 19 for BTC and Web, respectively. And we manually set $4 \leq k \leq 8$ and $15 \leq k \leq 23$ to test the effect.

The results show that for any two level number $k1$ and $k2$, where $k1 < k2$, G_{k2} is denser than that of G_{k1}. The increase in density is largely due to the decrease in the vertex number, but the overall size (in terms of both edge number and vertex number) of G_{k2} is

	k	$\|V_{G_k}\|$	$\|E_{G_k}\|$	Label size	Indexing time (s)	Qtime-1 (ms)	Qtime-2 (ms)
BTC	4	232K	18.7M	3.8 GB	1422.10	3.59	0.06
BTC	5	167K	17.2M	4.9 GB	1574.19	4.77	0.05
BTC	6	134K	16.4M	7.1 GB	2057.98	6.31	0.04
BTC	7	114K	15.8M	11.2 GB	2147.93	7.83	0.04
BTC	8	100K	15.4M	18.1 GB	2711.81	9.09	0.04
Web	15	329K	17.8M	5.8 GB	1398.34	17.10	15.63
Web	17	280K	16.0M	7.1 GB	1729.28	19.57	13.65
Web	19	242K	14.5M	8.1 GB	2034.07	20.58	7.81
Web	21	211K	13.2M	9.0 GB	2437.78	23.43	7.15
Web	23	186K	12.2M	9.9 GB	2784.04	32.61	6.55

Table 3: Performance results of different k: the size of G_k and label, indexing time, Qtime-1 for the time taken to process the labels and Qtime-2 for the time to run bi-Dijsktra in G_k

smaller than that of G_{k1}. Thus, as k increases, bi-Dijkstra runs on a denser but smaller graph G_{k2}. As shown by Qtime-2 in Table 3, bi-Dijkstra is still faster in G_{k2} than in G_{k1}, though G_{k2} is denser.

The overall query time, i.e., the sum of Qtime-1 and Qtime-2, increases as k increases. This is mainly due to the fact that the label size increases when k increases, i.e., the number of levels in the vertex hierarchy increases. Thus, the time to retrieve the labels from disk (leading to high I/O cost) and to intersect the label sets, i.e., Qtime-1, becomes more expensive. However, we note that if k is too small, it becomes very costly to run bi-Dijkstra in G_k. For example, it takes 108.53 ms to run bi-Dijkstra in $G_0 = G$ for the Web graph, which is larger than the sum of Qtime-1 and Qtime-2 for any k shown in Table 3.

Finally, we can also see from the table that the value of k computed by our algorithm is effective. In fact, for the Web graph, $k = 19$ computed by our algorithm is the optimal value.

7. LIMITATIONS OF EXISTING WORK

We highlight the challenges of computing P2P distance by discussing existing approaches and their limitations.

7.1 Indexing Approaches

Cohen et al. [16] proposed the 2-hop labeling that computes for each vertex v two sets, $L_{in}(v)$ and $L_{out}(v)$, where for each vertex $u \in L_{in}(v)$ and $w \in L_{out}(v)$, there is a path from u to v and from v to w. The distances $dist_G(u, v)$ and $dist_G(v, w)$ are pre-computed. Given a distance query, s and t, the index ensures that $dist_G(s, t)$ can be answered as $\min_{v \in (L_{out}(s) \cap L_{in}(t))} \{dist_G(s, v) + dist_G(v, t)\}$. However, com-

puting the 2-hop labeling, including the heuristic algorithms [15, 34], is very costly for large graphs.

Xiao et al. [40] exploit symmetric structures in an unweighted undirected graph to compress BFS trees to answer distance queries. However, the overall size of all the compressed BFS trees is prohibitively large even for medium sized graphs.

Wei [38] proposed an index based on a tree decomposition of an undirected graph G, where each node in the tree stores a set of vertices in G. The distance between each pair of vertices stored in each tree node is pre-computed, so that queries can be answered by considering the minimum distance between vertices stored on a simple path in the tree. However, the number of vertices stored in the root node is large and thus the pair-wise distances for these vertices are expensive to compute and require huge storage space. As a result, the method cannot scale to handle large graphs.

Jin et al. [22] proposed to use a spanning tree as a highway structure in a directed graph, so that distance from s to t is computed as the length of the shortest path from s to some vertex u, then from u via the highway (i.e., a path in the spanning tree) to some vertex v, and finally from v to t. Every vertex is given a label so that a set of entry points in the highway (e.g., u) and a set of exit points (e.g., v) can be obtained. However, the labeling requires all pairs shortest paths to be computed and is too costly for processing large graphs.

Cheng et al. [13] proposed the VC-index for computing the distances from a source vertex to all other vertices. The use of vertex cover in [13] has inspired the use of independent set in our method. VC-index can be used to answer P2P distance queries if we force the search from source s to stop when hitting target t. However, their method is not a labeling technique and much computation will be wasted in computing the distances from the source to many irrelevant vertices. There is also another method that uses vertex cover but is for answering k-hop reachability queries [14].

The problem of P2P distance querying has been well studied for road networks. Abraham et al. [2] recently proposed a hub-based labeling algorithm, which is the fastest known algorithm in the road network setting. Other fast algorithms such as [31], [17], and [9] are also based on the concept of highways to reduce the search space. However, it has been shown in [4] and [1] that the effectiveness of these methods relies on properties such as low VC dimensions and low highway dimensions, which are typical in road networks but may not hold for other types of graphs. Another approach is based on a concise representation of all pairs shortest paths [30, 32]. However, this approach heavily depends on the spatial coherence of vertices and their inter-connectivity. Tao et al. [35] proposed the k-skip shortest path on road networks, which is to return at least one out of every k consecutive vertices in a shortest path. While P2P distance querying has been quite successfully resolved for road networks, these methods are in general not applicable to graphs from other sources.

7.2 Other Approaches

When the input graph is too large to fit in main memory, external memory algorithms can be used to reduce the high disk I/O cost. Existing external memory algorithms are mainly for computing single-source shortest paths [23, 27, 25, 26] or BFS [6, 7, 11, 24, 28], which are wasteful for computing P2P distance.

There are also a number of approximation methods [8, 19, 29, 33, 36] proposed to compute P2P distance. Although these methods have a lower complexity than the exact methods in general, they are still quite costly for processing large graphs, in terms of both preprocessing time and storage space. We focus on exact distance querying but remark that approximation can be applied on top of our method (e.g., on the graph G_k defined in Section 4).

7.3 Methods with Similar Characteristics

We also want to specifically compare and contrast with several methods for road-networks that share some similar characteristics with IS-Label.

First, we consider the landmark approach proposed in [18], in which a set of landmarks L are selected and for each vertex, the distance to and from each landmark $x \in L$ is pre-computed. By triangle inequality, $dist_G(s,t)$ is lower bounded by $\max_{x \in L}\{|dist_G(s,x) - dist_G(t,x)|\}$, which helps speed up bi-Dijkstra search. With the full vertex hierarchy in IS-Label, we also store pre-computed distances in labels, except that the labels can always give the exact value of $dist_G(s,t)$ instead of a lower bound.

Next, we compare with the multilevel overlay graphs in [21] for road networks, where overlay graphs are built on top of shrinking sets S of vertices, preserving the distances among vertices in S. The overlay graphs can limit the search in a graph smaller than G. However, in building overlay graph for a vertex set S, each vertex in S undergoes a Dijkstra's single source shortest path search in G. The complexity is high and the empirical studies in [21] handled graphs with less than 500,000 vertices. While all overlay graphs are maintained in [21] for querying, we do not keep any graph for full vertex hierarchy and keep only G_k for k-level hierarchy.

Labeling technique is also used in the hub-based labeling algorithm (HL) [2], which is the fastest known algorithm for distance querying in road networks. As pointed out by the same authors in [3], for graphs other than road networks, the contraction hierarchy (CH) preprocessing in HL is not effective for general graphs. To address this problem they proposed hierarchical hub labeling (HHL) for other graph types [3]. Instead of the bottom up CH method, a top-down method was proposed for HHL, as inspired by [16]. HHL maintains a shortest path tree for every vertex to represent all uncovered shortest paths starting at each vertex. The computation and storage complexities of HHL are not scalable for handling large graphs. With a RAM of 96GB, HHL was tested only on preferential attachment graphs and small world graphs with $|V| = 100,000$ and average degree 10. In contrast, our labeling technique accommodates I/O efficient algorithms that work with small RAM sizes, such as 4GB, and we can handle preferential attachment graphs and small world graphs with $|V| = 1,000,000$ and average degree 80.

8. PATH QUERIES

In this section, we discuss the extension of our method to answer shortest-path queries. To answer a P2P shortest-path query, we need to keep some extra information in the vertex labels. When an augmenting edge (u,w) is created in G_i with $\omega_{G_i}(u,w) = \omega_{G_{i-1}}(u,v) + \omega_{G_{i-1}}(v,w)$, we also keep the intermediate vertex v along with the augmenting edge to indicate that the edge represents the path $\langle u,v,w\rangle$. Note that (u,v) and (v,w) are edges in G_{i-1}, which in turn can be augmenting edges. In the labeling process, instead of adding the entry $(w,d(u,w))$ to $label(u)$, we also attach the intermediate vertex v (if any) for (u,w). Thus, the entry becomes a triple $(w,d(u,w),v)$ (or $(w,d(u,w),\phi)$, if there is no intermediate vertex). Note that we keep the graph G_k, and thus the intermediate vertex of any augmenting edge in G_k is directly attached to the edge.

Given a query, s and t, if the query is of Type 1, the answer is determined by two label entries, $(w,d(s,w),v)$ and $(w,d(t,w),v')$. If $v \neq \phi$ (similarly for v'), we form two new queries (s,v) and (v,w). In this way, we recursively form queries until the intermediate vertex in a label entry is ϕ. It is then straightforward to obtain the resulting path by linking all the intermediate vertices. If the

query is of Type 2, then the answer is determined by two label entries and a path in G_k. The subpaths from the two label entries are derived in the same way as we do for a Type 1 query. The path in G_k is expanded into the original path in G by forming new queries, "u and v" and "v and w", for any augmenting edge (u, w) with the intermediate vertex v. For each such query, the corresponding subpath is obtained as discussed above. The I/O complexity of the overall process is given by $O(|SP_G(s,t)|)$, where $|SP_G(s,t)|$ is the number of edges on $SP_G(s,t)$.

9. CONCLUSIONS

In this paper, we introduced an effective disk-based indexing method named IS-Label for distance and shortest path querying in large real-world graphs, both directed and undirected. There are two major ideas in our approach. Firstly, we developed an independent-set based vertex hierarchy and proved that it can guide the labeling process of vertices. Secondly, for massive graphs, we can limit the height of the hierarchy so that the label size can be controlled and our theoretical analysis show that based on the labels, we need only one in-memory bi-Dijkstra's search on the residual graph to obtain the final solution.

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